Sato-Tate groups of abelian threefolds

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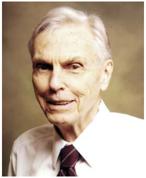


John Tate

Joint work with Francesc Fité and Kiran Kedlaya

Sato-Tate in dimension 1

Let $E/\mathbb{Q}$ be an elliptic curve, say,

$$y^2 = x^3 + Ax + B,$$

and let p be a prime of good reduction (so $p \nmid \Delta(E)$).

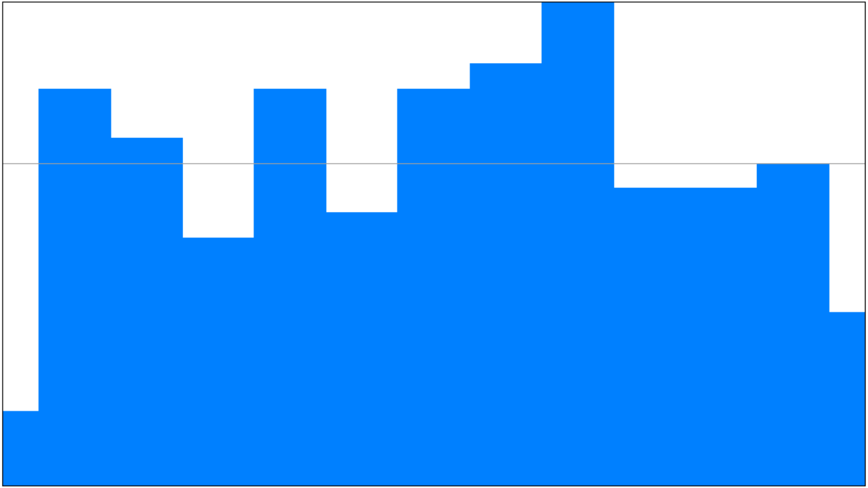
The number of $\mathbb{F}_p$-points on the reduction E_p of E modulo p is

$$\#E_p(\mathbb{F}_p) = p + 1 - t_p,$$

where the trace of Frobenius t_p is an integer in $[-2\sqrt{p},\, 2\sqrt{p}]$.

We are interested in the limiting distribution of $x_p := -t_p/\sqrt{p} \in [-2, 2]$ as p varies over primes of good reduction up to $N \to \infty$.

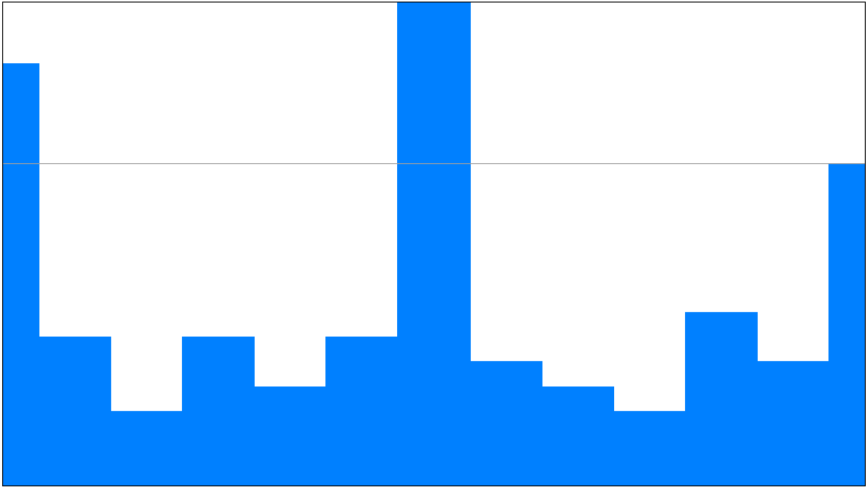
a1 histogram of y^2 = x^3 + x + 1 for p <= 2^10
170 data points in 13 buckets, z1 = 0.029, out of range data has area 0.018



Moments: 1 0.051 1.039 0.081 2.060 0.294 4.971 1.134 13.278 4.308 37.954

(click histogram to animate with Adobe reader or click here to view the animation in your browser)

a1 histogram of y^2 = x^3+1 for p <= 2^10
170 data points in 13 buckets, z1 = 0.518, out of range data has area 0.418



Moments: 1 -0.044 0.934 -0.160 2.754 -0.660 9.051 -2.655 31.232 -10.427 110.831

(click histogram to animate with Adobe reader or click here to view the animation in your browser)

Sato-Tate distributions in dimension 1

1. Typical case (no CM)

Elliptic curves $E/\mathbb{Q}$ w/o CM have the semi-circular trace distribution. (Also known for E/k, where k is a totally real or CM number field).

[CHT08, Taylor08, HST10, BGG11, BGHT11, ACCGHHNSTT18]

2. Exceptional cases (CM)

Elliptic curves E/k with CM have one of two distinct trace distributions, depending on whether k contains the CM field or not.

[Hecke, Deuring, early 20th century]

Sato-Tate groups in dimension 1

The Sato-Tate group of E is a closed subgroup G of $\mathrm{SU}(2) = \mathrm{USp}(2)$ that is determined by the ℓ-adic Galois representation attached to E.

A refinement/generalization of the Sato-Tate conjecture states that the distribution of normalized Frobenius traces of E converges to the distribution of traces in its Sato-Tate group G (under its Haar measure).

G	G/G^0	E	k	$\mathrm{E}[x_p^0], \mathrm{E}[x_p^2], \mathrm{E}[x_p^4] \ldots$
$\mathrm{SU}(2)$	C_1	$y^2 = x^3 + x + 1$	$\mathbb{Q}$	$1, 1, 2, 5, 14, 42, \ldots$
$N(\mathrm{U}(1))$	C_2	$y^2 = x^3 + 1$	$\mathbb{Q}$	$1, 1, 3, 10, 35, 126, \ldots$
$\mathrm{U}(1)$	C_1	$y^2 = x^3 + 1$	$\mathbb{Q}(\sqrt{-3})$	$1, 2, 6, 20, 70, 252, \ldots$

Fun fact: in the non-CM case the Sato-Tate conjecture implies that $E[x_p^n] = \frac{1}{2\pi} \int_0^{\pi} (2\cos\theta)^n \sin^2\theta \, d\theta$ is the $\frac{n}{2}$th Catalan number.

L-polynomials of Abelian varieties

Let A be an abelian variety over a number field k and fix a prime ℓ. The action of $\mathrm{Gal}(\bar{k}/k)$ on the ℓ-adic Tate module

$$V_\ell(A) := \varprojlim A[\ell^n] \otimes_{\mathbb{Z}} \mathbb{Q}$$

gives rise to the ℓ-adic Galois representation

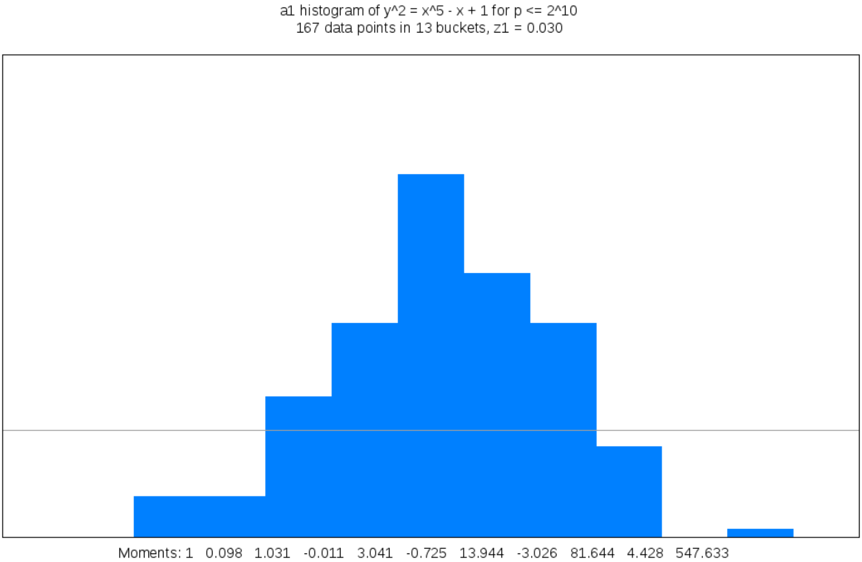
$$\rho_\ell \colon \mathrm{Gal}(\bar{k}/k) \to \mathrm{Aut}_{\mathbb{Q}_\ell}(V_\ell(A)) \simeq \mathrm{GSp}_{2g}(\mathbb{Q}_\ell).$$

For each prime $\mathfrak{p}$ of good reduction for A we have the L-polynomial

$$L_\mathfrak{p}(T) := \det(1 - \rho_\ell(\mathrm{Frob}_\mathfrak{p})T), \qquad \bar{L}_\mathfrak{p}(T) := L_\mathfrak{p}(T/\sqrt{\|\mathfrak{p}\|}),$$

which appears as an Euler factor in the L-series

$$L(A,s) := \prod_\mathfrak{p} L_\mathfrak{p}(\|\mathfrak{p}\|^{-s})^{-1}.$$



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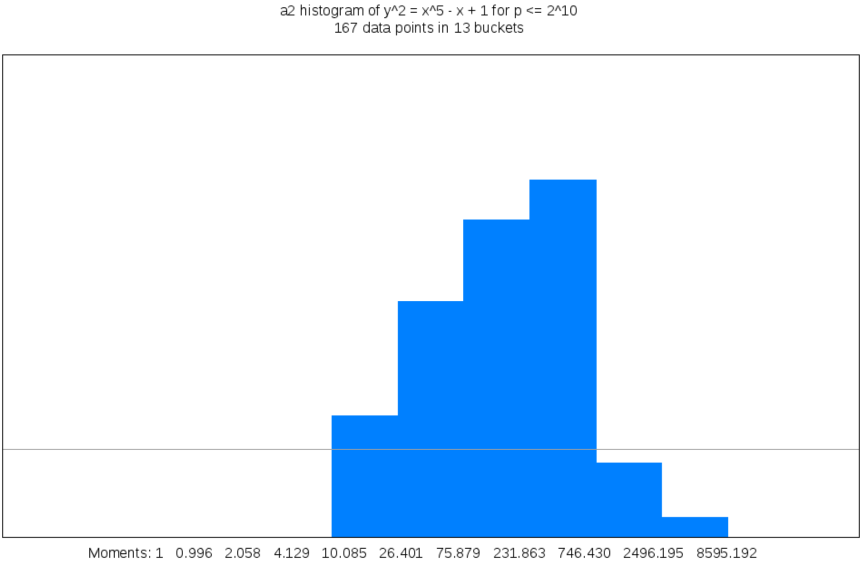
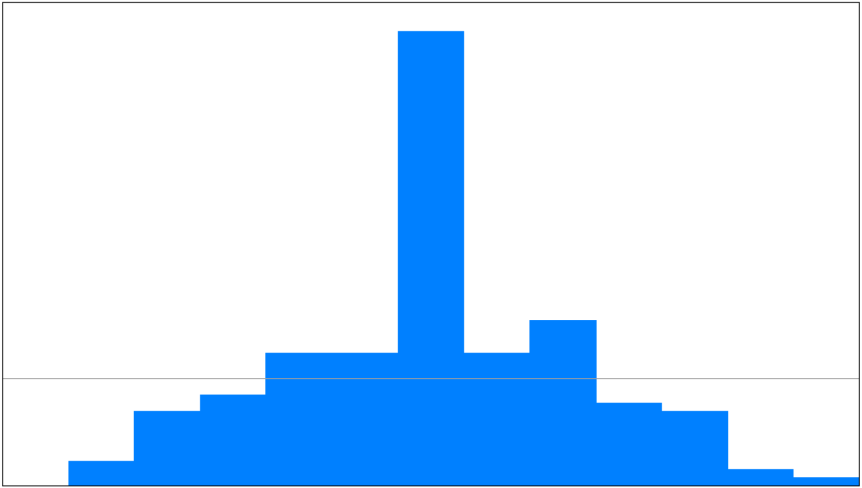


(click histogram to animate with Adobe reader or click here to view the animation in your browser)

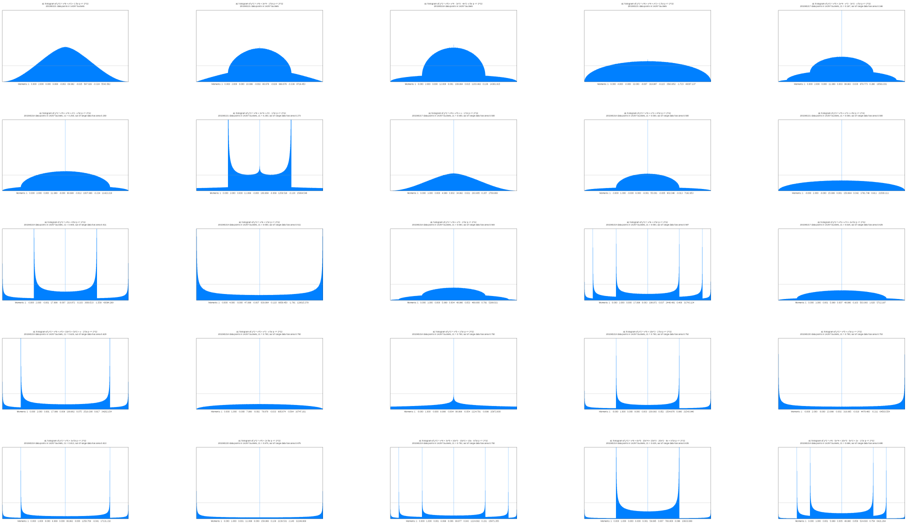
a1 histogram of y^2 = x^5 + 2x^4 - x^3 - 3x^2 - x for p <= 2^10
168 data points in 13 buckets, z1 = 0.196



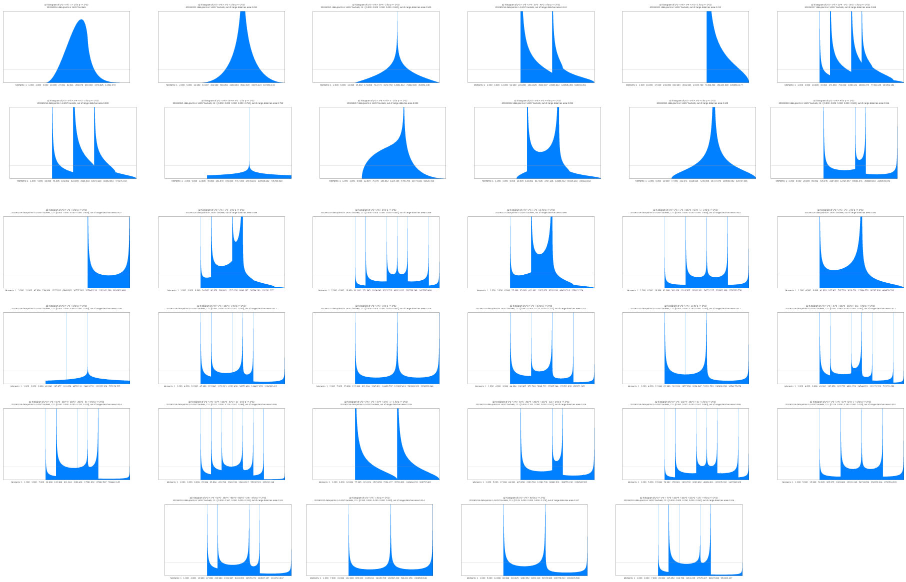
Moments: 1 0.034 1.822 0.225 9.597 4.081 71.210 68.943 658.625 1080.045 7157.897

(click histogram to animate with Adobe reader or click here to view the animation in your browser)

Exceptional a_1 distributions of abelian surfaces over $\mathbb{Q}$



a_2 distributions of abelian surfaces over $\mathbb{Q}$



The Sato-Tate group of an abelian variety

The Zariski closure of the image of ℓ-adic representation

$$\rho_\ell \colon G_k \to \mathrm{Aut}_{\mathbb{Q}_\ell}(V_\ell(A)) \simeq \mathrm{GSp}_{2g}(\mathbb{Q}_\ell)$$

is the ℓ-adic monodromy group $G_\ell \subseteq \mathrm{GSp}_{2g}$ of A, a $\mathbb{Q}_\ell$-algebraic group.
Fix $\iota \colon \mathbb{Q}_\ell \hookrightarrow \mathbb{C}$ and let $G^1_{\ell,\iota} := (G_\ell \cap \mathrm{Sp}_{2g}) \otimes_\iota \mathbb{C}$, a $\mathbb{C}$-algebraic group.

Definition [Serre]

$\mathrm{ST}(A) \subseteq \mathrm{USp}(2g)$ is a maximal compact subgroup of $G^1_{\ell,\iota}(\mathbb{C})$ equipped with $s \colon \mathfrak{p} \mapsto \mathrm{conj}(\|\mathfrak{p}\|^{-1/2}\rho_{\ell,\iota}(\mathrm{Frob}_\mathfrak{p})) \in \mathrm{Conj}(\mathrm{ST}(A))$, inducing $\mathfrak{p} \mapsto \bar{L}_\mathfrak{p}(T)$.

Sato-Tate conjecture for abelian varieties.

The conjugacy classes $s(\mathfrak{p})$ are equidistributed with respect to $\mu_{\mathrm{ST}(A)}$, the pushforward of the Haar measure to $\mathrm{Conj}(\mathrm{ST}(A))$.

Sato-Tate axioms for abelian varieties

$G \subseteq \mathrm{USp}(2g)$ satisfies the Sato-Tate axioms (for abelian varieties) if:

1. **Compact**: G is closed;
2. **Hodge**: G contains a Hodge circle $\theta\colon \mathrm{U}(1) \to G^0$ whose elements $\theta(u)$ have eigenvalues u, $1/u$ with multiplicity g, such that the conjugates of θ conjugates generate a dense subset of G;
3. **Rationality**: for each component H of G and each irreducible character χ of $\mathrm{GL}_{2g}(\mathbb{C})$ we have $\mathrm{E}[\chi(\gamma) : \gamma \in H] \in \mathbb{Z}$;
4. **Lefschetz**: The subgroup of $\mathrm{USp}(2g)$ fixing $\mathrm{End}(\mathbb{C}^{2g})^{G_0}$ is G^0.

Theorem [FKRS12, FKS19]

The Sato-Tate group $\mathrm{ST}(A)$ satisfies the Sato-Tate axioms if $g \leq 3$.

Axioms 1-3 are expected to hold in general, but Axiom 4 fails for $g = 4$.
For any g, the set of G satisfying axioms 1-3 is **finite**.

Galois endomorphism types

Let A be an abelian variety defined over a number field k.
Let K be the minimal extension of k for which $\mathrm{End}(A_K) = \mathrm{End}(A_{\bar{k}})$.
$\mathrm{Gal}(K/k)$ acts on the $\mathbb{R}$-algebra $\mathrm{End}(A_K)_{\mathbb{R}} = \mathrm{End}(A_K) \otimes_{\mathbb{Z}} \mathbb{R}$.

Definition

The *Galois endomorphism type* of A is the isomorphism class of $[\mathrm{Gal}(K/k), \mathrm{End}(A_K)_{\mathbb{R}}]$, where $[G, E] \simeq [G', E']$ iff there are isomorphisms $G \simeq G'$ and $E \simeq E'$ compatible with the group actions.

Theorem [FKRS12]

For abelian varieties A/k of dimension $g \leq 3$ there is a one-to-one correspondence between Sato-Tate groups and Galois types.

More precisely, the identity component G^0 is uniquely determined by $\mathrm{End}(A_K)_{\mathbb{R}}$ and $G/G^0 \simeq \mathrm{Gal}(K/k)$ (with corresponding actions).

Real endomorphism algebras of abelian surfaces

abelian surface	$\mathrm{End}(A_K)_{\mathbb{R}}$	$\mathrm{ST}(A)^0$
generic abelian surface	$\mathbb{R}$	$\mathrm{USp}(4)$
• RM abelian surface • product of non-CM elliptic curves	$\mathbb{R}\times\mathbb{R}$	$\mathrm{SU}(2)\times\mathrm{SU}(2)$
product of CM and non-CM elliptic curves	$\mathbb{C}\times\mathbb{R}$	$\mathrm{U}(1)\times\mathrm{SU}(2)$
• CM abelian surface • product of CM elliptic curves	$\mathbb{C}\times\mathbb{C}$	$\mathrm{U}(1)\times\mathrm{U}(1)$
• QM abelian surface • square of non-CM elliptic curve	$\mathrm{M}_2(\mathbb{R})$	$\mathrm{SU}(2)_2$
square of CM elliptic curve	$\mathrm{M}_2(\mathbb{C})$	$\mathrm{U}(1)_2$

(factors in products are assumed to be non-isogenous)

Sato-Tate groups of abelian surfaces

Theorem [FKRS12]

Up to conjugacy in $\mathrm{USp}(4)$, 55 groups satisfy the Sato-Tate axioms, of which 3 cannot arise as Sato-Tate groups of abelian surfaces.

Theorem [FKRS12]

Up to conjugacy in $\mathrm{USp}(4)$ there are 52 Sato-Tate groups of abelian surfaces over number fields, of which 9 are maximal.

The 9 maximal groups all arise as the Sato-Tate group of a genus 2 curve defined over $\mathbb{Q}$; the rest can be realized via base change.

There are 6 distinct identity components that arise, and the order of each component group divides $48 = 2^4 \cdot 3$.

Note: This theorem says nothing about equidistribution, however this is now known in many special cases [FS12, Johansson13, Taylor18].

Maximal Sato-Tate groups of abelian surfaces

$\mathbf{ST}(A)^0$	$\mathbf{ST}(A)/\mathbf{ST}(A)^0$	$A/\mathbb{Q}$
$\mathrm{USp}(4)$	C_1	$\mathrm{Jac}(y^2 = x^5 - x + 1)$
$\mathrm{SU}(2)^2$	C_2	$\mathrm{Jac}(y^2 = x^6 + x^5 + x - 1)$
$\mathrm{U}(1) \times \mathrm{SU}(2)$	C_2	$\mathrm{Jac}(y^2 = x^6 + 3x^4 - 2)$
$\mathrm{U}(1)^2$	D_2	$\mathrm{Jac}(y^2 = x^6 + 3x^4 + x^2 - 1)$
	C_4	$\mathrm{Jac}(y^2 = x^5 + 1)$
$\mathrm{SU}(2)_2$	D_4	$\mathrm{Jac}(y^2 = x^5 + x^3 + 2x)$
	D_6	$\mathrm{Jac}(y^2 = x^6 + x^3 - 2)$
$\mathrm{U}(1)_2$	$\mathrm{D}_6 \times \mathrm{C}_2$	$\mathrm{Jac}(y^2 = x^6 + 3x^5 + 10x^3 - 15x^2 + 15x - 6)$
	$\mathrm{S}_4 \times \mathrm{C}_2$	$\mathrm{Jac}(y^2 = x^6 - 5x^4 + 10x^3 - 5x^2 + 2x - 1)$

Each of the 9 maximal Sato-Tate groups in dimension 2 can be realized by the Jacobian of a genus 2 curve over $\mathbb{Q}$.

There are 3 subgroups of $N(\mathrm{U}(1) \times \mathrm{U}(1))$ that satisfy the Sato-Tate axioms but do not occur as Sato-Tate groups of abelian surfaces.

Sato-Tate groups of abelian threefolds

Theorem [FKS19]

Up to conjugacy in $\mathrm{USp}(6)$, 433 groups satisfy the Sato-Tate axioms, of which 23 cannot arise as Sato-Tate groups of abelian threefolds.

Theorem [FKS19]

Up to conjugacy in $\mathrm{USp}(6)$ there are 410 Sato-Tate groups of abelian threefolds over number fields, of which 33 are maximal.

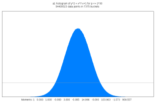
The 33 maximal groups all arise as the Sato-Tate group of an abelian threefold defined over $\mathbb{Q}$; the rest can be realized via base change.

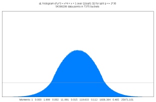
There are 14 distinct identity components that arise, and the order of every component group always divides one of the following integers: $192 = 2^6 \cdot 3, \;\; 336 = 2^4 \cdot 3 \cdot 7, \;\; 432 = 2^4 \cdot 3^3$.

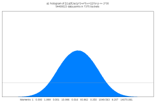
Real endomorphism algebras of abelian threefolds

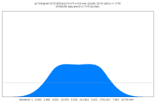
abelian threefold	$\mathbf{End}(A_K)_{\mathbb{R}}$	$\mathbf{ST}(A)^{\mathbf{0}}$
generic abelian threefold	$\mathbb{R}$	$\mathrm{USp}(6)$
quadratic CM abelian threefold	$\mathbb{C}$	$\mathrm{U}(3)$
product of a non-CM elliptic curve and an abelian surface	$\mathbb{R} \times \mathbb{R}$	$\mathrm{SU}(2) \times \mathrm{USp}(4)$
product of a CM elliptic curve and an abelian surface	$\mathbb{C} \times \mathbb{R}$	$\mathrm{U}(1) \times \mathrm{USp}(4)$
• RM abelian threefold • product of a non-CM elliptic curve and an RM abelian surface • product of 3 non-CM elliptic curves	$\mathbb{R} \times \mathbb{R} \times \mathbb{R}$	$\mathrm{SU}(2)^3$
• product of a CM elliptic curve and an RM abelian surface • product of a CM elliptic curve and two non-CM elliptic curves	$\mathbb{C} \times \mathbb{R} \times \mathbb{R}$	$\mathrm{U}(1) \times \mathrm{SU}(2)^2$
• product of a CM abelian surface and a non-CM elliptic curve • product of two CM elliptic curves and a non-CM elliptic curve	$\mathbb{C} \times \mathbb{C} \times \mathbb{R}$	$\mathrm{U}(1)^2 \times \mathrm{SU}(2)$
• CM abelian threefold • product of a CM elliptic curve and a CM abelian surface • product of three CM elliptic curves	$\mathbb{C} \times \mathbb{C} \times \mathbb{C}$	$\mathrm{U}(1)^3$
• product of non-CM elliptic curve and a QM abelian surface • product of a non-CM elliptic curve and the square of a non-CM elliptic curve	$\mathbb{R} \times \mathrm{M}_2(\mathbb{R})$	$\mathrm{SU}(2) \times \mathrm{SU}(2)_2$
• product of a CM elliptic curve and a QM abelian surface • product of a CM elliptic curve and the square of non-CM elliptic curve	$\mathbb{C} \times \mathrm{M}_2(\mathbb{R})$	$\mathrm{U}(1) \times \mathrm{SU}(2)_2$
product of a non-CM elliptic curve and the square of CM elliptic curve	$\mathbb{R} \times \mathrm{M}_2(\mathbb{C})$	$\mathrm{SU}(2) \times \mathrm{U}(1)_2$
product of a CM elliptic curve and the square of CM elliptic curve	$\mathbb{C} \times \mathrm{M}_2(\mathbb{C})$	$\mathrm{U}(1) \times \mathrm{U}(1)_2$
cube of a non-CM elliptic curve	$\mathrm{M}_3(\mathbb{R})$	$\mathrm{SU}(2)_3$
cube of a CM elliptic curve	$\mathrm{M}_3(\mathbb{C})$	$\mathrm{U}(1)_3$

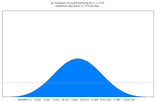
Connected Sato-Tate groups of abelian threefolds

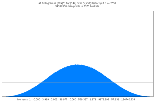

$\mathrm{USp}(6)$

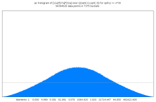

$\mathrm{U}(3)$

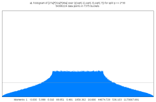

$\mathrm{SU}(2) \times \mathrm{USp}(4)$

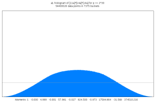

$\mathrm{U}(1) \times \mathrm{USp}(4)$

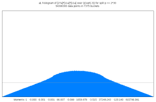

$\mathrm{SU}(2)^3$

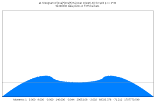

$\mathrm{U}(1) \times \mathrm{SU}(2)^2$

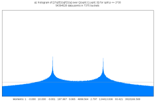

$\mathrm{U}(1)^2 \times \mathrm{SU}(2)$

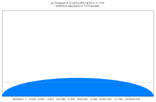

$\mathrm{U}(1)^3$

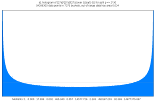

$\mathrm{SU}(2) \times \mathrm{SU}(2)_2$


$\mathrm{U}(1) \times \mathrm{SU}(2)_2$


$\mathrm{SU}(2) \times \mathrm{U}(1)_2$


$\mathrm{U}(1) \times \mathrm{U}(1)_2$


$\mathrm{SU}(2)^2$


$\mathrm{U}(1)^3$

Maximal Sato-Tate groups of abelian threefolds

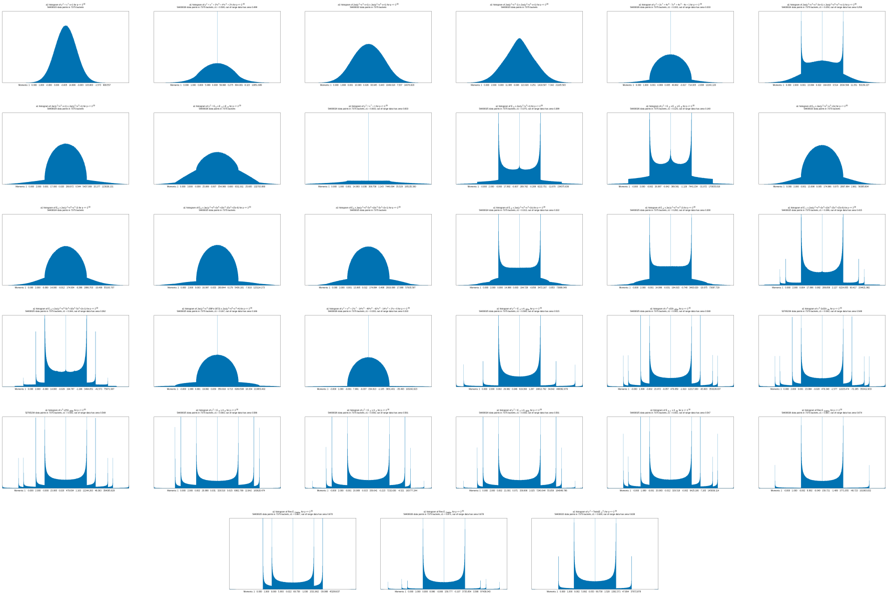
$\mathrm{ST}(A)^0$	$\mathrm{ST}(A)/\mathrm{ST}(A)^0$	$A/\mathbb{Q}$
$\mathrm{USp}(4)$	$\langle 1,1\rangle$	$\mathrm{Jac}(y^2 = x^7 - x + 1)$
$\mathrm{U}(3)$	$\langle 2,1\rangle$	$\mathrm{Jac}(y^2 = x^7 + 3x^5 + 4x^3 + 2x)$
$\mathrm{SU}(2)\times\mathrm{USp}(4)$	$\langle 1,1\rangle$	$\mathrm{Jac}(y^2 = 4x^8 - 7x^2 + 4)$
$\mathrm{U}(1)\times\mathrm{USp}(4)$	$\langle 2,1\rangle$	$\mathrm{Jac}(y^2 = x^8 - x^6 - 3x^4 + x^2 - 1)$
$\mathrm{SU}(2)^3$	$\langle 6,1\rangle$	$\mathrm{Jac}(y^2 = 2x^7 + 4x^6 - 7x^4 + 4x^3 - 4x + 1)$
$\mathrm{U}(1)\times\mathrm{SU}((2)^2$	$\langle 4,2\rangle$	$\mathrm{Jac}(y^2 = x^8 + 2x^6 + 4x^4 + 4x^2 + 4)$
$\mathrm{U}(1)^2\times\mathrm{SU}(2)$	$\langle 4,1\rangle$	$\mathrm{Jac}(y^2 = x^8 - 5x^6 + 10x^4 - 10x^2 + 5)$
	$\langle 4,2\rangle$	$\mathrm{Jac}(y^2 = x^8 + 4x^6 - 2x^4 + 4x^2 + 1)$
$\mathrm{U}(1)^3$	$\langle 6,2\rangle$	$\mathrm{Jac}(y^2 = x^7 - 1)$
	$\langle 8,2\rangle$	$\mathrm{Jac}(y^2 = x^8 - 8x^6 + 20x^4 - 16x^2 + 2)$
	$\langle 8,5\rangle$	$\mathrm{Jac}(3x^4 + 2y^4 + 6z^4 - 6x^2y^2 + 6x^2z^2 - 12y^2z^2 = 0)$
$\mathrm{SU}(2)\times\mathrm{SU}(2)_2$	$\langle 8,3\rangle$	$\mathrm{Jac}(y^2 = x^8 + x^4 + 2)$
	$\langle 12,4\rangle$	$\mathrm{Jac}(y^2 = x^8 + 8x^6 + 18x^4 + 16x^2 - 4)$
$\mathrm{U}(1)\times\mathrm{SU}(2)_2$	$\langle 16,11\rangle$	$\mathrm{Jac}(y^2 = x^8 + 2x^6 + 4x^2 - 4)$
	$\langle 24,14\rangle$	$\mathrm{Jac}(2y^4 + 4x^2y^2 - 6y^2z^2 + 6x^3z + xz^3 + 3z^4 = 0)$
$\mathrm{SU}(2)\times\mathrm{U}(1)_2$	$\langle 24,14\rangle$	$\mathrm{Jac}(x^4 - 8x^2y^2 + 24y^4 + 24xy^2z - 12xz^3 - 9z^4 = 0)$
	$\langle 48,48\rangle$	$\mathrm{Jac}(x^4 - y^4 + 2x^2z^2 + xz^3 = 0)$
$\mathrm{U}(1)\times\mathrm{U}(1)_2$	$\langle 48,51\rangle$	`49a1` $\times\, \mathrm{Jac}(y^2 = x^6 + 3x^5 + 10x^3 - 15x^2 + 15x - 6)$
	$\langle 96,226\rangle$	`27a1` $\times\, \mathrm{Jac}(y^2 = x^6 - 5x^4 + 10x^3 - 5x^2 + 2x - 1)$

Maximal Sato-Tate groups of abelian threefolds

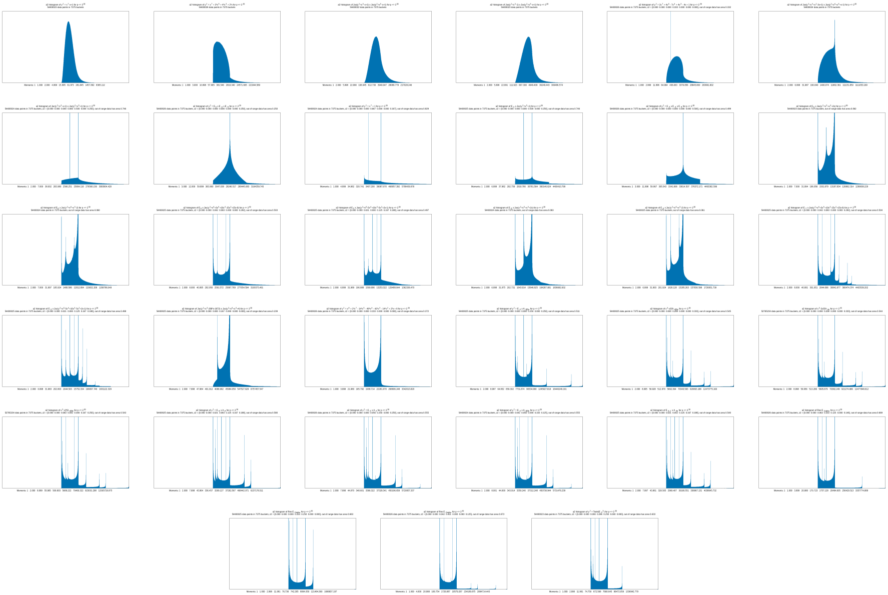
$\mathrm{ST}(A)^0$	$\mathrm{ST}(A)/\mathrm{ST}(A)^0$	$A/\mathbb{Q}$
$\mathrm{SU}(2)_3$	$\langle 12, 4\rangle$	$\mathrm{Jac}(x^3z + 4y^3z + 3x^2y^2 - z^4)$
	$\langle 24, 12\rangle$	$\mathrm{Jac}(y^2 = x^8 + 2x^7 - 14x^5 - 49x^4 - 42x^3 - 14x^2 + 2x + 6)$
$\mathrm{U}(1)_3$	$\langle 48, 15\rangle$	27a1 $\times\ \mathrm{Twist}_{\langle 24,10\rangle}($27a1$)$
	$\langle 48, 15\rangle$	32a2 $\times\ \mathrm{Twist}_{\langle 24,1\rangle}($32a2$)$
	$\langle 48, 38\rangle$	$\mathrm{Jac}(y^2 = x^8 - x^7 - 7x^6 - 7x^5 - 35x^4 - 35x^3 - 21x^2 + 3x + 6)$
	$\langle 48, 41\rangle$	32a2 $\times\ \mathrm{Twist}_{\langle 24,5\rangle}($32a2$)$
	$\langle 96, 193\rangle$	256a1 $\times\ \mathrm{Jac}(y^2 = x^6 - 5x^4 + 10x^3 - 5x^2 + 2x - 1)$
	$\langle 144, 125\rangle$	27a1 $\times\ \mathrm{Twist}_{\langle 72,25\rangle}($27a1$)$
	$\langle 144, 127\rangle$	27a1 $\times\ \mathrm{Twist}_{\langle 72,25\rangle}($27a1$)$
	$\langle 192, 988\rangle$	64a4 $\times\ \mathrm{Prym}(y^4 = x^4 + 2x^2 + x)$
	$\langle 192, 956\rangle$	$\mathrm{Res}(y^2 = x^3 - \alpha x) \qquad [\alpha^3 - \alpha^2 + \alpha - 2 = 0]$
	$\langle 336, 208\rangle$	$\mathrm{Jac}(2x^3y - 2x^3z - 3x^2z^2 - 2xy^3 - 2xz^3 - 4y^3z + 3y^2z^2 - yz^3 = 0)$
	$\langle 432, 523\rangle$	$\mathrm{Res}(y^2 = x^3 - \alpha) \qquad [\alpha^3 - \alpha^2 + \alpha - 2 = 0]$
	$\langle 432, 734\rangle$	$\mathrm{Twist}_{\langle 648,533\rangle}($27a1$)$

For $A = \mathrm{Twist}_{\langle n,i\rangle}(E)$ we have $L(A,s)L(A',s) = L(E_M,\rho,s)$, where $E/\mathbb{Q}$ has CM by M and ρ is an Artin representation with $\rho(G_M) \simeq \langle n, i\rangle$. We then have $A_{\overline{\mathbb{Q}}} \sim E_{\overline{\mathbb{Q}}}^{\dim\rho}$ and $A_M \sim A'_M$, with $\mathrm{ST}(A) = \mathrm{ST}(A')$.

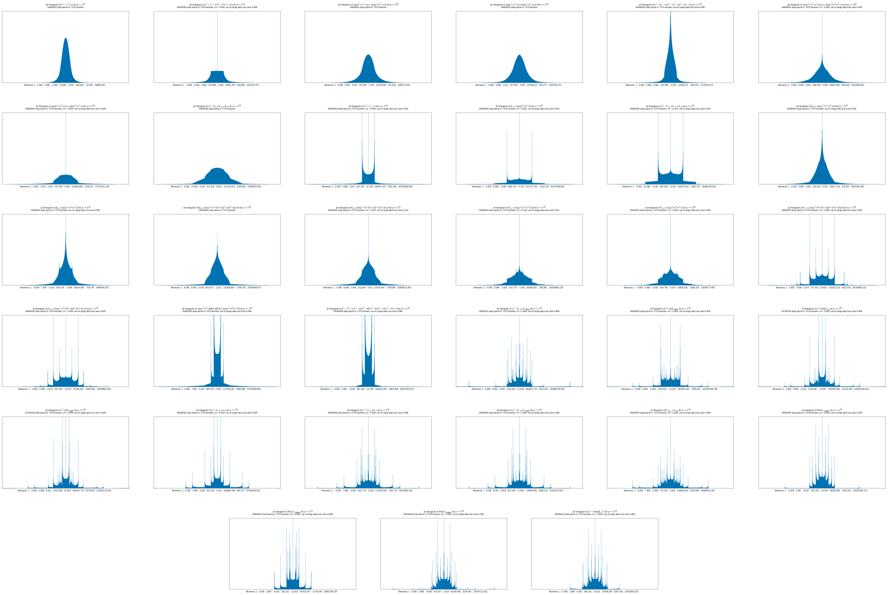
a_1 distributions of maximal Sato-Tate groups for $g = 3$



a_2 distributions of maximal Sato-Tate groups for $g = 3$



a_3 distributions of maximal Sato-Tate groups for $g = 3$



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