Computing the endomorphism ring of an ordinary elliptic curve

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Elliptic curves

An *elliptic curve* E/F is a smooth projective curve of genus 1 with a distinguished rational point 0.

The set $E(F)$ of rational points on E form an abelian group.

For $\mathrm{char}(F) \neq 2, 3$ we define E with an affine equation

$$y^2 = x^3 + Ax + B,$$

where $4A^3 + 27B^2 \neq 0$. The *j-invariant* of E is

$$j(E) = 12^3 \frac{4A^3}{4A^3 + 27B^2}.$$

If $F = \overline{F}$ then $j(E)$ uniquely identifies E (but not in $\mathbb{F}_q$).

Elliptic curves over finite fields

Consider $F = \mathbb{F}_q$. The size of the group $E(\mathbb{F}_q)$ is

$$\#E(\mathbb{F}_q) = q + 1 - t,$$

for some integer t with $|t| \le 2\sqrt{q}$. The SEA algorithm computes t in polynomial time (very fast in practice).

Typically t is nonzero in $\mathbb{F}_q$, in which case E is called *ordinary*.

Some useful facts about $t = t(E)$:

1. $t(E_1) = t(E_2) \Longleftrightarrow E_1$ and E_2 are isogenous.
2. $j(E_1) = j(E_2)$ and $t(E_1) = t(E_2) \iff E_2 \cong E_2$.
3. $j(E_1) = j(E_2) \Longrightarrow |t(E_1)| = |t(E_2)| \quad$ for $j(E_1) \notin \{0, 12^3\}$.

Maps between elliptic curves

An *isogeny* $\phi : E_1 \to E_2$ is a rational map (defined over $\overline{F}$) with $\phi(0) = 0$. It induces a homomorphism from $E_1(F)$ to $E_2(F)$.

The *endomorphism ring* $\mathrm{End}(E)$ contains all $\phi : E \to E$.
We have $\mathbb{Z} \subseteq \mathrm{End}\, E$, but for $F = \mathbb{F}_q$, equality never holds.

If $E/\mathbb{F}_q$ is ordinary, then $\mathrm{End}(E) \cong \mathcal{O}(D)$ where

$$\mathcal{O}(D) = \mathbb{Z} + \frac{D + \sqrt{D}}{2}\mathbb{Z}$$

is the imaginary quadratic order of some discriminant D.

We want to compute D.

The Frobenius endomorphism

The endomorphism $\pi : (x, y) \rightsquigarrow (x^q, y^q)$ on $E(\overline{\mathbb{F}}_q)$ satisfies

$$\pi^2 - t\pi + q = 0.$$

If we set $D_\pi = t^2 - 4q$ and fix an isomorphism $\mathrm{End}\, E \cong \mathcal{O}(D)$ we may regard $\pi = \frac{t+\sqrt{D_\pi}}{2}$ as an element of $\mathcal{O}(D)$.

Thus $\mathcal{O}(D_\pi) \subseteq \mathcal{O}(D)$, which implies $D|D_\pi$ and that D and D_π have the same fundamental discriminant D_K.

By factoring $D_\pi = v^2 D_K$ we may determine D_K and v.
We then have $D = u^2 D_K$ for some $u|v$.

We want to compute u.

This is easy if v is small (or smooth), but may be hard if not.

Computing isogenies

We call a (separable) isogeny ϕ an ℓ-isogeny if $\#\ker\phi = \ell$.
We restrict to prime ℓ, in which case $\ker\phi$ is cyclic.

The classical modular polynomial $\Phi_\ell \in \mathbb{Z}[X, Y]$ has the property

$$\Phi_\ell\big(j(E_1), j(E_2)\big) = 0 \quad \Longleftrightarrow \quad E_1 \text{ and } E_2 \text{ are } \ell\text{-isogenous.}$$

The ℓ-isogeny graph $G_\ell(\mathbb{F}_q)$ has vertex set

$$\mathcal{E}(\mathbb{F}_q) = \{j(E/\mathbb{F}_q)\} = \mathbb{F}_q,$$

and edges (j_1, j_2) for $\Phi_\ell(j_1, j_2) = 0$ (note Φ_ℓ is symmetric).

Φ_ℓ is big: $O(\ell^{3+\epsilon})$ bits.

The structure of the ℓ-isogeny graph [Kohel]

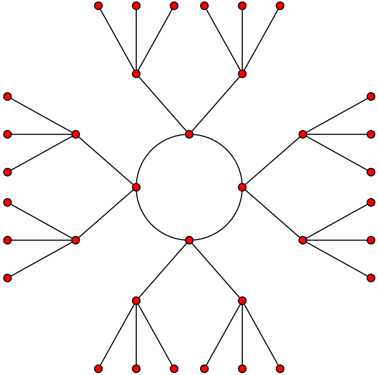
The connected components of $G_\ell(\mathbb{F}_q)$ are ℓ-*volcanoes*.
An ℓ-volcano of height h has vertices in level $V_0, \ldots, V_h$.

Vertices in V_0 have endomorphism ring $\mathcal{O}(D_0)$ with $\ell \nmid u_0$.
Vertices in V_k have endomorphism ring $\mathcal{O}(\ell^{2k} D_0)$.

1. The subgraph on V_0 is a cycle (the *surface*).
   All other edges lie between V_k and V_{k+1} for some k.
2. For $k > 0$ each vertex in V_k has one neighbor in V_{k-1}.
3. For $k < h$ every vertex in V_k has degree $\ell + 1$.

See [Kohel 1996], [Fouquet-Morain 2002], or [S 2009] for more details.

A 3-volcano of height 2 with a 4-cycle



Algorithms to compute u

- **Isogeny climbing**: computes ℓ-isogenies for prime $\ell | v$ to determine the power of ℓ dividing u in.
  Probabilistic complexity $O(q^{3/2+\epsilon})$.
- **Kohel's algorithm**: computes the kernel of n-isogenies, where $n = O(q^{1/6})$ need not be a divisor of v.
  Deterministic complexity $O(q^{1/3+\epsilon})$ (GRH).
- **New algorithm**: computes the cardinality of smooth relations using isogenies of subexponential degree.
  Probabilistic complexity $L[1/2, \sqrt{3}/2](q)$ (GRH+).

$$L[\alpha, c](x) = \exp\left((c + o(1))(\log x)^{\alpha}(\log\log x)^{1-\alpha}\right).$$

All algorithms have unconditionally correct output.

The action of the class group [CM theory]

For an invertible ideal $\mathfrak{a} \subset \mathcal{O}_D \cong \mathrm{End}(E)$, let $E[\mathfrak{a}]$ be the subgroup of points annihilated by all $a \in \mathfrak{a}$. The map

$$j(E) \to j(E/E[\mathfrak{a}])$$

corresponds to an isogeny of degree $N(\mathfrak{a})$.

This defines a group action by the ideal group on the set

$$\{j(E/\mathbb{F}_q) : \mathrm{End}(E) \cong \mathcal{O}(D)\}.$$

This action factors through the class group $\mathrm{cl}(\mathcal{O}(D)) = \mathrm{cl}(D)$. The action is faithful and transitive.

See the books of [Cox], [Lang], or [Silverman] for more on CM theory.

Walking isogeny cycles

If $\ell \nmid v$ and $\left(\frac{D}{\ell}\right) = 1$, the ℓ-volcano containing $j(E)$ is a cycle of length $|\alpha|$, where $\alpha \in \mathrm{cl}(D)$ contains an ideal of norm ℓ.

We can compute $|\alpha|$ (without knowing D) by walking a path $j_0, j_1, \ldots$ in $G_\ell(\mathbb{F}_q)$ starting from $j_0 = j(E)$:

1. Let j_1 be one of the two roots of $\Phi_\ell\big(X, j_0\big)$ in $\mathbb{F}_q$.
2. Let j_{k+1} be the unique root of $\Phi_\ell\big(X, j_k\big)/(X - j_{k-1})$ in $\mathbb{F}_q$.

The choice of j_1 is arbitrary (we cannot distinguish α and α^{-1}). In either case, $|\alpha|$ (and $|\alpha^{-1}|$) is the least n for which $j_n = j_0$.

Step 2 finds the unique root of a degree ℓ polynomial $f(X)$ over $\mathbb{F}_q$. Complexity is $T(\ell) = O(\ell^2 + \mathrm{M}(\ell) \log q)$ operations in $\mathbb{F}_q$.

Computing End(E) with class groups (naïvely)

Given $E/\mathbb{F}_q$, let $\#E = q + 1 - t$ and $4q = t^2 - v^2 D_K$, so that End(E) $\cong \mathcal{O}(D)$ where $D = u^2 D_K$ for some $u|v$.

If $u_1, \ldots, u_m$ are the divisors of v, then $u = u_i$ for some i.

Pick any $\ell \nmid v$ satisfying $\left(\frac{D_K}{\ell}\right) = 1$.

For each $D_i = u_i^2 D_K$ there is an element $\alpha_i \in \mathrm{cl}(D_i)$ containing an ideal of norm ℓ, but $|\alpha_i|$ typically varies with i.

We can compare $|\alpha_i|$ to the length of the ℓ-isogeny cycle containing $j(E)$. These must be equal if $u = u_i$.

This is too slow, but we can exploit this idea.

Relations

A *relation* R is a pair of vectors $(\ell_1, \ldots, \ell_k)$ and $(e_1, \ldots, e_k)$.

We say R *holds* in $\mathrm{cl}(D)$ if for each i there is an $\alpha_i \in \mathrm{cl}(D)$ containing an ideal of norm ℓ_i such that

$$\alpha_1^{e_1} \cdots \alpha_k^{e_k} = 1.$$

More generally, we define the *cardinality* of R in $\mathrm{cl}(D)$ by

$$\#R/D = \#\left\{\tau \in \{\pm 1\}^k : \prod \alpha_i^{\tau_i e_i} = 1 \text{ in } \mathrm{cl}(D)\right\}.$$

$\#R/D$ does not depend on the choice of α_i.

Counting relations

Given a relation R with $(\ell_1, \ldots, \ell_k)$ and $(e_1, \ldots, e_k)$:

1. Set J_0 be a list containing the single element $j(E)$.
2. For each element in J_i walk e_i steps in both directions of the ℓ_i cycle and append the two end points to the list J_{i+1}.
3. $\#R/E$ is the number of times $j(E)$ appears in the list J_k.

The complexity is $\sum_{i=1}^{k} 2^i e_i T(\ell_i)$ operations in $\mathbb{F}_q$.

The key lemma

Lemma: If $\mathcal{O}(D_1) \subseteq \mathcal{O}(D_2)$ then $\#R/D_1 \leq \#R/D_2$.

Proof: There is a norm-preserving map from $\mathcal{O}(D_1)$ to $\mathcal{O}(D_2)$ that induces a group homomorphism from $\mathrm{cl}(D_1)$ to $\mathrm{cl}(D_2)$.

Corollary: Let $p \parallel v$ and set $D_1 = (v/p)^2 D_K$ and $D_2 = p^2 D_K$.
Let R be a relation with $\#R/D_1 > \#R/D_2$.
If u is the conductor of $\mathcal{O}(D) \cong \mathrm{End}(E)$ then

$$p|u \quad \Longleftrightarrow \quad \#R/E < \#R/D_1.$$

Theorem: Such an R exists.

Conjecture: Almost all R that hold in $\mathrm{cl}(D_1)$ don't hold in $\mathrm{cl}(D_2)$.

Algorithm to compute End(E)

Given $E/\mathbb{F}_q$, the following algorithm computes $D = u^2 D_K$, the discriminant of the order isomorphic to End(E).

1. Compute $t = q + 1 - \#E$, v, and D_k, with $4q = t^2 - v^2 D_K$.
2. For primes $p|v$, find a relation R satisfying the corollary. Count $\#R/E$ in the isogeny graph to test whether $p|u$.
3. Output $u^2 D_K$.

The algorithm above assumes v is square-free.

Finding smooth relations

The following algorithm is adapted from Hafner/McCurley.

We seek a smooth relation in $\mathrm{cl}(D_1)$.
Pick a smoothness bound B and a small constant k_0 (say 3).

1. Let $\ell_1, \ldots, \ell_n$ be the primes up to B with $\left(\frac{D_1}{\ell_i}\right) = 1$, and let $\alpha_i \in \mathrm{cl}(D_1)$ contain an ideal of norm ℓ_i.
2. Generate $\beta = \prod \alpha_i^{x_i}$ where all but k_0 of the x_i are zero and the other x_i are suitably bounded.
3. For each β, test whether $N(b)$ is B-smooth, where b is a the reduced representative of β.
4. If so write $\prod \alpha_i^{x_i} = \prod \alpha_i^{y_i}$ and compute R.
   Verify that $\#R/D_1 > \#R/D_2$ (almost always true).

For suitable B, the complexity is $L[1/2, \sqrt{3}/2](|D|)$

An example of cryptographic size (200 bits)

We have $4q = t^2 - v^2 D_K$ where $t = 212$, $D_K = -7$ and

$$v = 2 \cdot 127 \cdot \underbrace{524287}_{p_1} \cdot \underbrace{7195777666870732918103}_{p_2}.$$

After finding $2 \nmid u$ and $127 \nmid u$ we test $p_1 | u$ by computing

$$R_1 = (2^{2533}, 11^{752}, 29^2, 37^{47}, 79^1, 113^1, 149^1, 151^2, 347^1, 431^1),$$

which holds in $\mathrm{cl}(p_2^2 D_K)$ but not $\mathrm{cl}(p_1^2 D_K)$.We test $p_2 | u$ using

$$R_2 = (2^{23}, 11^5, 43^1, 71^2),$$

which holds in $\mathrm{cl}(p_1^2 D_K)$ but not in $\mathrm{cl}(p_2^2 D_K)$.

Total time to compute $\mathrm{End}(E)$ is under 30 minutes

Certifying the endomorphism ring

To verify a claimed value of u, it suffices to have a relation R_p for each prime divisor of v such that:

1. For each prime $p|(v/u)$, we have $\#R_p/E > \#R_p/p^2D_K$.
2. For each prime $p|u$, we have $\#R_p/(u/p)^2D_K > \#R_p/E$.

Certificate size is $O(\log^{2+\epsilon} q)$.

Note that either $D_1 u^2 D_K$ or $D_1 = (u/p)^2 D_K$.
We always have $D_1 \leq D$. Very useful when $D \ll D_\pi$.

This yields an algorithm to compute u with complexity

$$L[1/2 + o(1), 1](|D|) \ + \ L[1/3, c](q)$$

which depends primarily on D, not q.