On the computation and evaluation of modular polynomials

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Isogenies of elliptic curves

An *elliptic curve* E/k is a smooth projective curve of genus 1 with a distinguished k-rational point 0.

An *isogeny* $\phi\colon E_1 \to E_2$ is a morphism of elliptic curves, a rational map that fixes the point 0. We shall assume $\phi \neq 0$.

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The induced homomorphism $\phi\colon E_1(\bar{k}) \to E_2(\bar{k})$ has a finite kernel. Conversely, every finite subgroup of $E_1(\bar{k})$ is the kernel of an isogeny.

The *degree* of an isogeny is its degree as a rational map. For nonzero *separable* isogenies, $\deg \phi = |\ker \phi|$.

We are primarily interested in isogenies of prime degree $\ell \neq \operatorname{char} k$, which are necessarily separable isogenies with cyclic kernels.

j-invariants

The $\bar{k}$-isomorphism classes of elliptic curves E/k are in bijection with the field k. For $E\colon\ y^2 = x^3 + Ax + B$, the j-*invariant* of E is

$$j(E) = j(A,B) = 1728\frac{4A^3}{4A^3 + 27B^2} \in k.$$

The j-invariants $j(0,B) = 0$ and $j(A,0) = 1728$ are special.
They correspond to elliptic curves with extra automorphisms.

For $j \notin \{0, 1728\}$, we have $j = j(A,B)$, where

$$A = 3j(1728 - j) \qquad \text{and} \qquad B = 2j(1728 - j)^2.$$

Note that $j(E_1) = j(E_2)$ does not necessarily imply that E_1 and E_2 are isomorphic over k, only that they are isomorphic over $\bar{k}$.

The modular equation

Let $j\colon \mathbb{H} \to \mathbb{C}$ be the classical modular function.

For any $\tau \in \mathbb{H}$, the values $j(\tau)$ and $j(\ell\tau)$ are the j-invariants of elliptic curves $E_\tau/\mathbb{C}$ and $E_{\ell\tau}/\mathbb{C}$ that are ℓ-isogenous.

The minimal polynomial $\Phi_\ell(Y)$ of the function $j(\ell z)$ over $\mathbb{C}(j)$ has coefficients that are integer polynomials in $j(z)$.

Replacing $j(z)$ with X yields the *modular polynomial* $\Phi_\ell \in \mathbb{Z}[X,Y]$ that parameterizes pairs of ℓ-isogenous elliptic curves $E/\mathbb{C}$:

$$\Phi_\ell\big(j(E_1), j(E_2)\big) = 0 \quad \Longleftrightarrow \quad j(E_1) \text{ and } j(E_2) \text{ are } \ell\text{-isogenous.}$$

This moduli interpretation remains valid over any field whose characteristic is not equal to ℓ.

$\Phi_\ell(X,Y) = 0$ is a defining equation for the affine modular curve $Y_0(\ell) = \Gamma_0(\ell)\backslash\mathbb{H}$.

Isogenies make hard problems easier

Isogenies play a key role in many applications:

- The Schoof-Elkies-Atkin (SEA) point-counting algorithm.
- Computing the endomorphism ring of an elliptic curve.
- The elliptic curve discrete logarithm problem (?).
- Computing Hilbert class polynomials $H_D(X)$.
- Computing modular polynomials.

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Modular polynomials $\Phi_\ell(X, Y)$ are used in all of these applications.

Given an elliptic curve E/F, the roots of the univariate polynomial

$$\phi_\ell(Y) = \Phi_\ell(j(E), Y) \in F[Y]$$

that lie in F are precisely the j-invariants of the elliptic curves $\tilde{E}/F$ that are ℓ-isogenous to E.

Modular polynomials are very large…

$\Phi_\ell \in \mathbb{Z}[X, Y]$ is symmetric, with degree $\ell + 1$ in both X and Y.
Asymptotically, its size is $O(\ell^3 \log \ell)$ bits.

ℓ	coefficients	largest	average	total
127	8258	7.5kb	5.3kb	5.5MB
251	31880	16kb	12kb	48MB
503	127262	36kb	27kb	431MB
1009	510557	78kb	60kb	3.9GB
2003	2009012	166kb	132kb	33GB
3001	4507505	259kb	208kb	117GB
4001	8010005	356kb	287kb	287GB
5003	12522512	454kb	369kb	577GB
10007	50085038	968kb	774kb	4.8TB

Size of $\Phi_\ell(X, Y)$

. . . but instantiated modular polynomials are not.

For an elliptic curve E over a finite field $\mathbb{F}_q$, the size of the instantiated polynomial $\phi_\ell(Y) = \Phi_\ell(j(E), Y)$ is only $O(\ell \log q)$ bits.

Even if q is quite large, say 4096 bits, for $\ell = 10007$ the size of $\phi_\ell(Y)$ is just 5MB, which is almost a million times smaller than $\Phi_\ell(X, Y)$.

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A quote from the former elliptic curve point-counting world record holder (at 2500 decimal digits):

"Despite this progress, computing modular polynomials remains the stumbling block for new point counting records. Clearly, to circumvent the memory problems, one would need an algorithm that directly obtains the polynomial specialised in one variable."

INRIA Project TANC, 2007

Results

Let $E/\mathbb{F}_q$ be an elliptic curve and let $\ell < q$ be a prime $(\ell \neq \operatorname{char} \mathbb{F}_q)$.

Theorem

Under the generalized Riemann hypothesis (GRH), one can compute the instantiated modular polynomial $\Phi_\ell(j(E), Y)$ using $O(\ell \log q)$ space in time quasi-linear in the size of Φ_ℓ (quasi-cubic in ℓ).

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Applying this to SEA, we can compute $\#E(\mathbb{F}_q)$ in $\tilde{O}(n^4)$ time and $O(n^2 \log n)$ space $(n = \log q)$, under standard heuristic assumptions. Previously, the SEA algorithm required $\Omega(n^3 \log n)$ space.

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The new algorithm is based on the *isogeny volcano* approach to computing modular polynomials [Bröker-Lauter-S 2012].

A volcano

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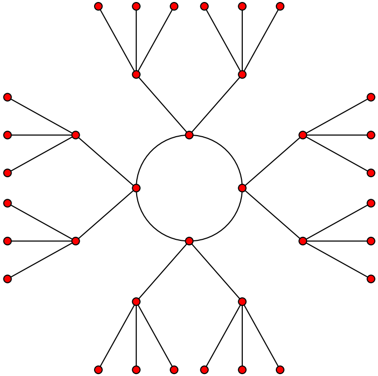


ℓ-volcanoes

For a prime ℓ, an ℓ-*volcano* is a connected undirected graph whose vertices are partitioned into levels $V_0, \ldots, V_d$ such that:

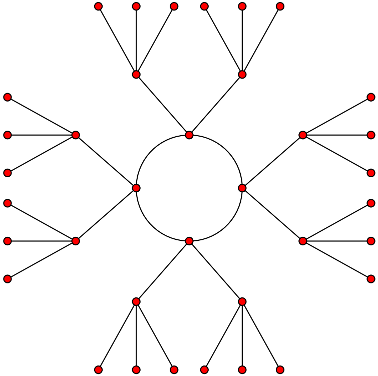
1. The subgraph on V_0 (the *surface*) is a connected regular graph of degree $0, 1$, or 2.

2. For $i > 0$, each $v \in V_i$ has exactly one neighbor in V_{i-1}. All edges not on the surface arise in this manner.

3. For $i < d$, each $v \in V_i$ has degree ℓ+1.

We allow self-loops and multi-edges, but this can happen only on the surface.

A 3-volcano of depth 2



The graph of ℓ-isogenies

Definition
The ℓ*-isogeny graph* $G_\ell(k)$ has vertex set $\{j(E) : E/k\} = k$
and edges (j_1, j_2) for each root $j_2 \in k$ of $\Phi_\ell(j_1, Y)$ (with multiplicity).

Except for $j \in \{0, 1728\}$, the in-degree of each vertex of G_ℓ is equal to its out-degree.

Thus G_ℓ is a bi-directed graph on $k \backslash \{0, 1728\}$, which we may regard as an undirected graph.

It consists of *ordinary* and *supersingular* components.

We have an infinite family of graphs $G_\ell(k)$ with vertex set k, one for each prime $\ell \neq \mathrm{char}(k)$.

An elliptic curve E over a field of characteristic $p > 0$ is supersingular iff $E[p] = \{0\}$.

Endomorphism rings

Isogenies from an elliptic curve E to itself are *endomorphisms*.
They form a ring $\mathrm{End}(E)$ under composition and point addition.

We always have $\mathbb{Z} \subseteq \mathrm{End}(E)$, due to scalar multiplication maps.
If $\mathbb{Z} \subsetneq \mathrm{End}(E)$, then E has *complex multiplication* (CM).

For an elliptic curve E with complex multiplication:

$$\mathrm{End}(E) \simeq \begin{cases} \text{order in an imaginary quadratic field} & \text{(ordinary),} \\ \text{order in a quaternion algebra} & \text{(supersingular).} \end{cases}$$

In characteristic $p > 0$, every elliptic curve has CM, since the p-power Frobenius endomorphism $(x, y) \mapsto (x^p, y^p)$ does not lie in $\mathbb{Z}$.

Horizontal and vertical isogenies

Let $\varphi\colon E_1 \to E_2$ by an ℓ-isogeny of ordinary elliptic curves with CM.
Let $\mathrm{End}(E_1) \simeq \mathcal{O}_1 = [1, \tau_1]$ and $\mathrm{End}(E_2) \simeq \mathcal{O}_2 = [1, \tau_2]$.

Then $\ell\tau_2 \in \mathcal{O}_1$ and $\ell\tau_1 \in \mathcal{O}_2$.

Thus one of the following holds:

- $\mathcal{O}_1 = \mathcal{O}_2$, in which case φ is *horizontal*;
- $[\mathcal{O}_1 : \mathcal{O}_2] = \ell$, in which case φ is *descending*;
- $[\mathcal{O}_2 : \mathcal{O}_1] = \ell$, in which case φ is *ascending*.

In the latter two cases we say that φ is a *vertical* isogeny.

The theory of complex multiplication

Let E/k have CM by an imaginary quadratic order $\mathcal{O}$.

For each invertible $\mathcal{O}$-ideal $\mathfrak{a}$, the $\mathfrak{a}$-torsion subgroup

$$E[\mathfrak{a}] = \{P \in E(\bar{k}) : \alpha(P) = 0 \text{ for all } \alpha \in \mathfrak{a}\}$$

is the kernel of an isogeny $\varphi_{\mathfrak{a}} : E \to E'$ of degree $N(\mathfrak{a}) = [\mathcal{O} : \mathfrak{a}]$.
We necessarily have $\operatorname{End}(E) \simeq \operatorname{End}(E')$, so $\varphi_{\mathfrak{a}}$ is **horizontal**.

If $\mathfrak{a}$ is principal, then $E' \simeq E$. This induces a $\operatorname{cl}(\mathcal{O})$-action on the set

$$\operatorname{Ell}_{\mathcal{O}}(k) = \{j(E) : E/k \text{ with } \operatorname{End}(E) \simeq \mathcal{O}\}.$$

This action is faithful and transitive; thus $\operatorname{Ell}_{\mathcal{O}}(k)$ is a principal homogeneous space, a *torsor*, for $\operatorname{cl}(\mathcal{O})$.

One can decompose horizontal isogenies of large prime degree into an equivalent sequence of isogenies of small prime degrees, which makes them **easy to compute**; see [Bröker-Charles-Lauter 2008, Jao-Souhkarev ANTS IX].

Isogeny volcanoes

Theorem (Kohel)

Let V be an ordinary connected component of $G_\ell(\mathbb{F}_q)$ that does not contain $0, 1728$. Then V is an ℓ-volcano in which the following hold:

(i) *Vertices in level V_i all have the same endomorphism ring $\mathcal{O}_i$.*

(ii) $\ell \nmid [\mathcal{O}_K : \mathcal{O}_0]$*, and* $[\mathcal{O}_i : \mathcal{O}_{i+1}] = \ell$.

(iii) *The subgraph on V_0 has degree $1 + (\frac{D}{\ell})$, where $D = \operatorname{disc}(\mathcal{O}_0)$.*

(iv) *If $(\frac{D}{\ell}) \geq 0$ then $|V_0|$ is the order of $[\mathfrak{l}]$ in $\operatorname{cl}(\mathcal{O}_0)$.*

(v) *The depth of V is $\operatorname{ord}_\ell(v)$, where $4q = t^2 - v^2 D$.*

The term *volcano* is due to Fouquet and Morain (ANTS V).
See `http://arxiv.org/abs/1208.5370` for more on isogeny volcanoes.

Modular polynomials via isogeny volcanoes [BLS]

Given an odd prime ℓ, we may compute $\Phi_\ell(X, Y)$ as follows:

1. Select a sufficiently large set of primes of the form $4p = t^2 - \ell^2 v^2 D$ with $\ell \nmid v$, $p \equiv 1 \bmod \ell$, and $h(D) > \ell + 1$.
2. For each prime p, compute $\Phi_\ell(X, Y) \bmod p$ as follows:
   a. Compute $\mathrm{Ell}_{\mathcal{O}}(\mathbb{F}_p)$ using $H_D(X) \bmod p$.
   b. Map the ℓ-volcanoes intersecting $\mathrm{Ell}_{O}(\mathbb{F}_p)$ (without using Φ_ℓ).
   c. Interpolate $\Phi_\ell(X, Y) \bmod p$.
3. Use the CRT to recover Φ_ℓ over $\mathbb{Z}$ (or mod q via the explicit CRT).

Under the GRH, the expected running time is $O(\ell^3 \log^{3+\epsilon} \ell)$ using $O(\ell^3 \log \ell)$ space (or $O(\ell^2 \log q)$ space to compute $\Phi_\ell \bmod q$).

We can similarly compute modular polynomials for other modular functions.
One can also use a CRT approach to compute Φ_N for composite N [Ono-S in prog].

Explicit Chinese Remainder Theorem

Suppose $c \equiv c_i \bmod p_i$ for k distinct primes p_i. Then

$$c \equiv \sum c_i a_i M_i \bmod M,$$

where $M = \prod p_i$, $M_i = M/p_i$ and $a_i = 1/M_i \bmod p_i$.
If $M > 2|c|$, we can recover $c \in \mathbb{Z}$.

Montgomery-Silverman 1990, Bernstein 1995, S 2011.

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With $M > 4|c|$, the explicit CRT computes $c \bmod q$ directly via

$$c = \left(\sum c_i a_i M_i - rM\right) \bmod q,$$

where $r = \operatorname{rnd}(\sum a_i c_i/p_i)$ is computed using $O(\log k)$ bits of precision.

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Using an online algorithm, this can be applied to N coefficients c in parallel, using $O(\log M + k \log q + N(\log q + \log k)) \approx O(N \log q)$ space.

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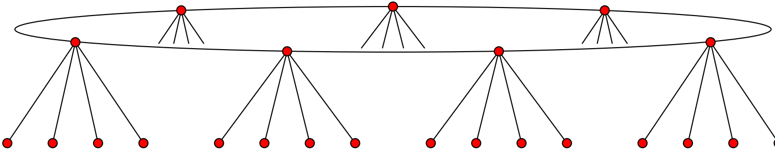
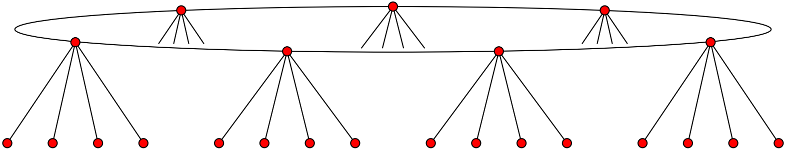
Mapping a volcano



Mapping a volcano

Example
$\ell \ = 5, \quad p = 4451, \quad D = -151$

General requirements
$4p = t^2 - v^2\ell^2 D, \quad p \equiv 1 \bmod \ell$



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$\ell = 5, \quad p = 4451, \quad D = -151$

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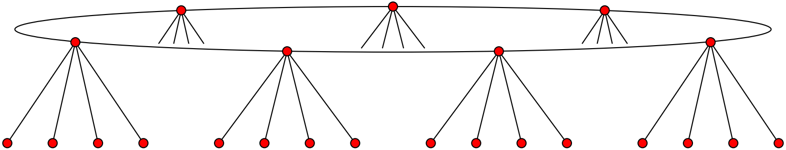

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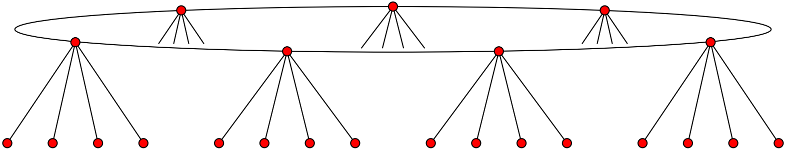

1. Find a root of $H_D(X)$

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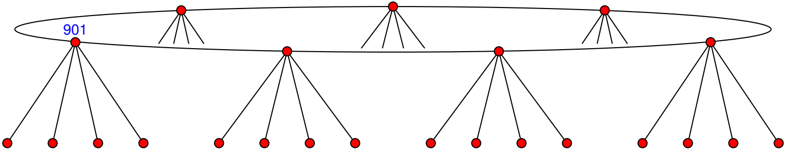

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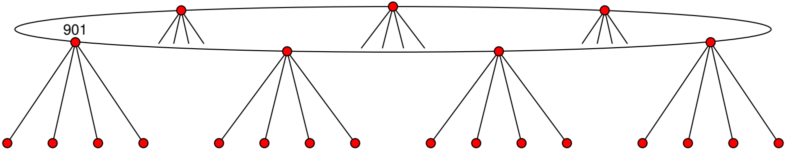

2. Enumerate surface using the action of α_{ℓ_0}

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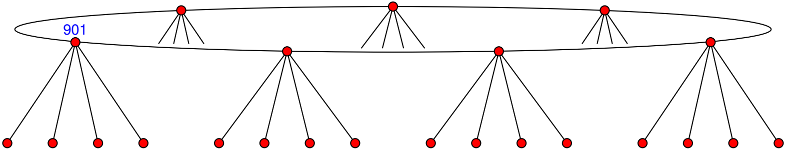

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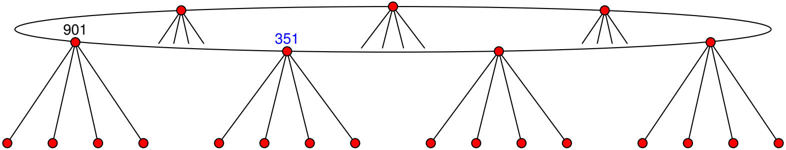

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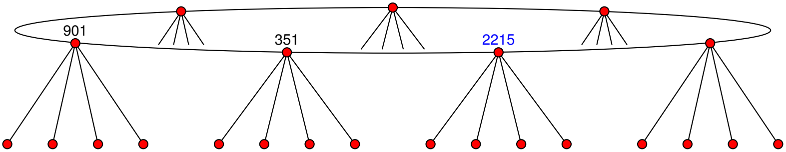

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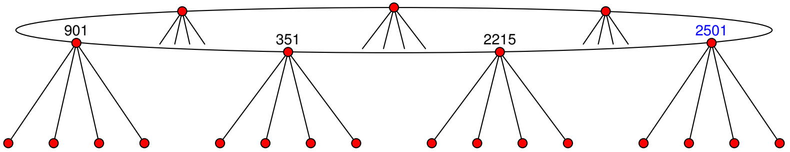

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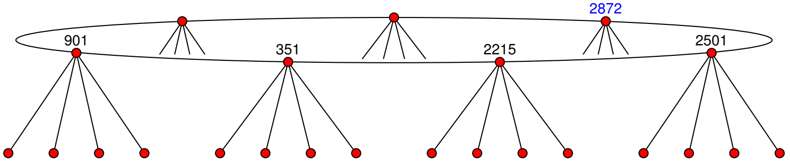

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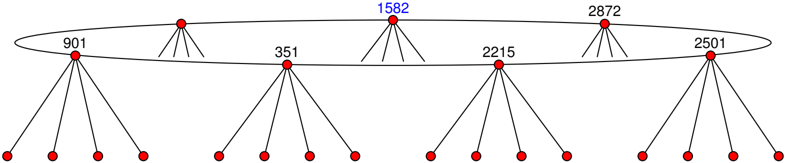
$\ell_0 = 2, \quad \alpha_5 = \alpha_2^3$

General requirements

$4p = t^2 - v^2\ell^2 D, \quad p \equiv 1 \bmod \ell$

$\ell \nmid v, \quad (\frac{D}{\ell}) = 1, \quad h(D) \geq \ell + 2$

$\ell_0 \neq \ell, \ (\frac{D}{\ell_0}) = 1, \ \alpha_\ell = \alpha_{\ell_0}^k$



2. Enumerate surface using the action of α_{ℓ_0}

$901 \xrightarrow{2} 1582 \xrightarrow{2} 2501 \xrightarrow{2} 351 \xrightarrow{2} 701 \xrightarrow{2} 2872 \xrightarrow{2} 2215 \xrightarrow{2}$

Mapping a volcano

Example

$\ell \;= 5, \quad p = 4451, \quad D = -151$

$t \;= 52, \quad v = 2, \quad h(D) = 7$

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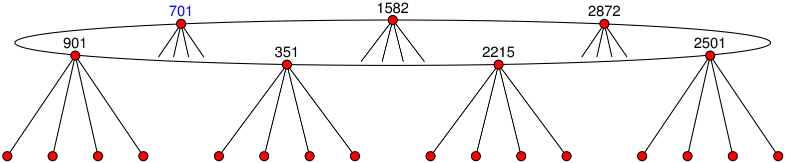

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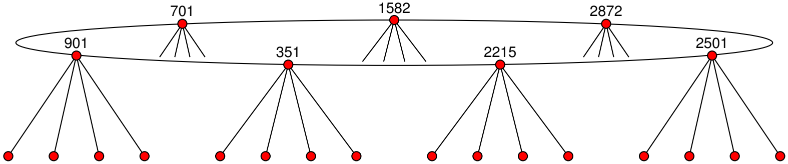

3. Descend to the floor using Vélu's formula

Mapping a volcano

Example

$\ell = 5, \quad p = 4451, \quad D = -151$

$t = 52, \quad v = 2, \quad h(D) = 7$

$\ell_0 = 2, \quad \alpha_5 = \alpha_2^3$

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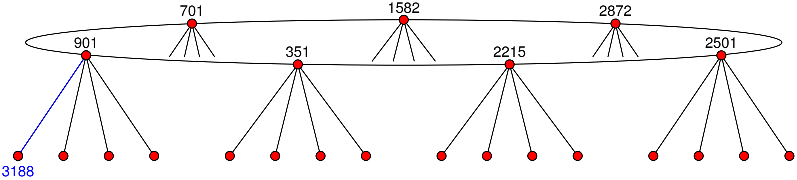

3. Descend to the floor using Vélu's formula: $901 \xrightarrow{5} 3188$

Mapping a volcano

Example

$\ell \ = 5, \quad p = 4451, \quad D = -151$

$t \ = 52, \quad v = 2, \quad h(D) = 7$

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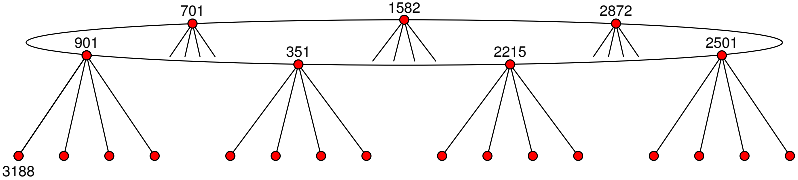

4. Enumerate floor using the action of β_{ℓ_0}

Mapping a volcano

Example

$\ell = 5, \quad p = 4451, \quad D = -151$

$t = 52, \quad v = 2, \quad h(D) = 7$

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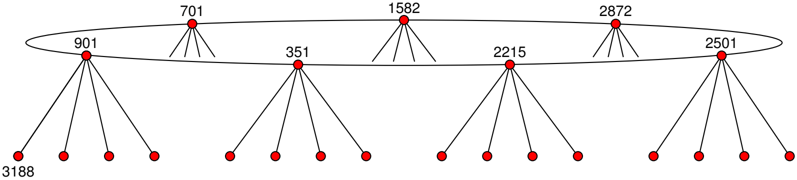

4. Enumerate floor using the action of β_{ℓ_0}

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Mapping a volcano

Example

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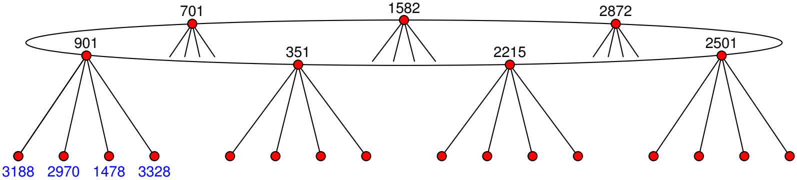

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Mapping a volcano

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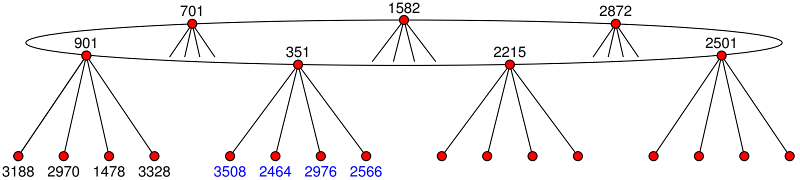

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Mapping a volcano

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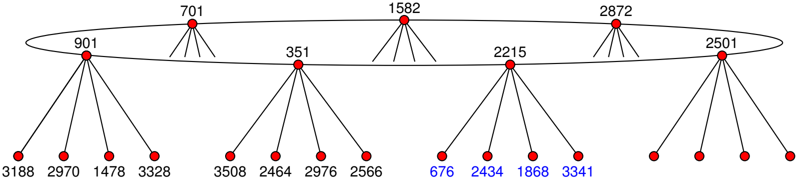

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Mapping a volcano

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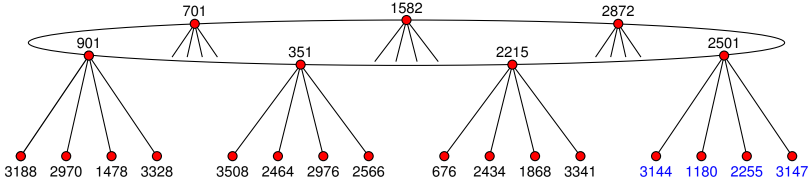

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$$
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Mapping a volcano

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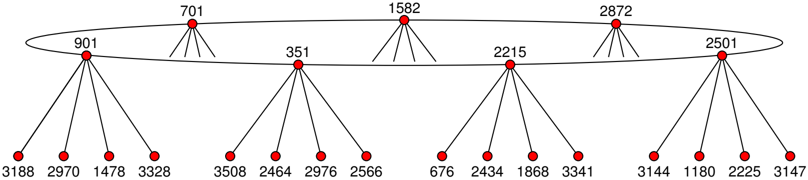

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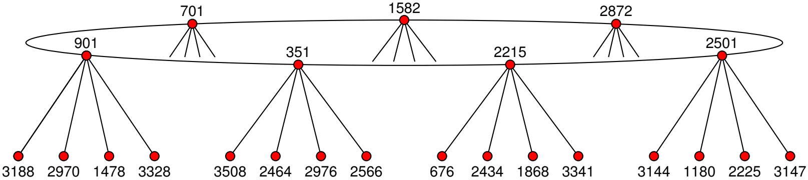

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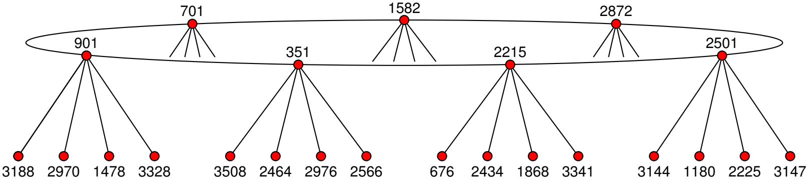

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Mapping a volcano

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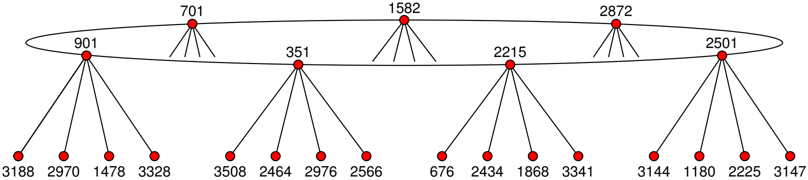
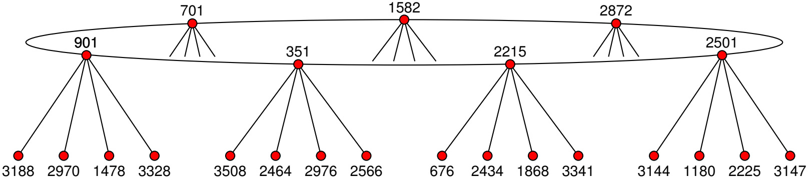
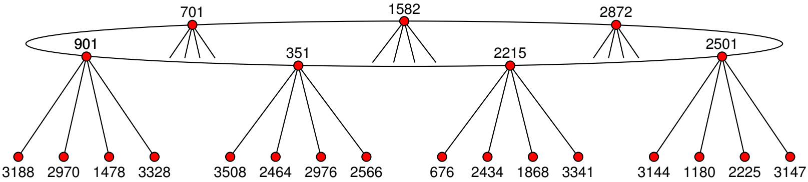


Interpolating $\Phi_\ell \bmod p$



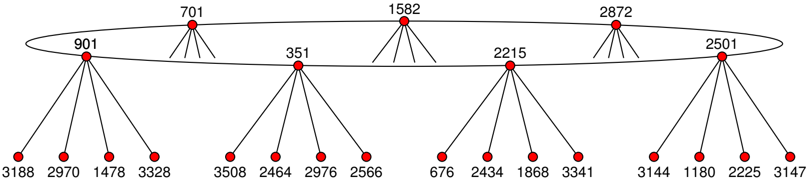
$$
\begin{aligned}
\Phi_5(X, 901) &= (X-701)(X-351)(X-3188)(X-2970)(X-1478)(X-3328)\\
\Phi_5(X, 351) &= (X-901)(X-2215)(X-3508)(X-2464)(X-2976)(X-2566)\\
\Phi_5(X, 2215) &= (X-351)(X-2501)(X-3341)(X-1868)(X-2434)(X-676)\\
\Phi_5(X, 2501) &= (X-2215)(X-2872)(X-3147)(X-2255)(X-1180)(X-3144)\\
\Phi_5(X, 2872) &= (X-2501)(X-1582)(X-1502)(X-4228)(X-1064)(X-2087)\\
\Phi_5(X, 1582) &= (X-2872)(X-701)(X-945)(X-3497)(X-3244)(X-291)\\
\Phi_5(X, 701) &= (X-1582)(X-901)(X-2843)(X-4221)(X-3345)(X-4397)
\end{aligned}
$$

Interpolating $\Phi_\ell \bmod p$



$$
\begin{aligned}
\Phi_5(X, 901) &= X^6 + 1337X^5 + 543X^4 + 497X^3 + 4391X^2 + 3144X + 3262\\
\Phi_5(X, 351) &= X^6 + 3174X^5 + 1789X^4 + 3373X^3 + 3972X^2 + 2932X + 4019\\
\Phi_5(X, 2215) &= X^6 + 2182X^5 + 512X^4 + 435X^3 + 2844X^2 + 2084X + 2709\\
\Phi_5(X, 2501) &= X^6 + 2991X^5 + 3075X^5 + 3918X^3 + 2241X^2 + 3755X + 1157\\
\Phi_5(X, 2872) &= X^6 + 389X^5 + 3292X^4 + 3909X^3 + 161X^2 + 1003X + 2091\\
\Phi_5(X, 1582) &= X^6 + 1803X^5 + 794X^4 + 3584X^3 + 225X^2 + 1530X + 1975\\
\Phi_5(X, 701) &= X^6 + 515X^5 + 1419X^4 + 941X^3 + 4145X^2 + 2722X + 2754
\end{aligned}
$$

Interpolating $\Phi_\ell \bmod p$



$$\begin{aligned}\Phi_5(X,Y) = X^6 &+ (4450Y^5 + 3720Y^4 + 2433Y^3 + 3499Y^2 + 70Y + 3927)X^5\\ &(3720Y^5 + 3683Y^4 + 2348Y^3 + 2808Y^2 + 3745Y + 233)X^4\\ &(2433Y^5 + 2348Y^4 + 2028Y^3 + 2025Y^2 + 4006Y + 2211)X^3\\ &(3499Y^5 + 2808Y^4 + 2025Y^3 + 4378Y^2 + 3886Y + 2050)X^2\\ &(70Y^5 + 3745Y^4 + 4006Y^3 + 3886Y^2 + 905Y + 2091)X\\ &(Y^6 + 3927Y^5 + 233Y^4 + 2211Y^3 + 2050Y^2 + 2091Y + 2108)\end{aligned}$$

The Weber function

The Weber $\mathfrak{f}$-function is defined by

$$\mathfrak{f}(\tau) = \frac{\eta\big((\tau+1)/2\big)}{\zeta_{48}\eta(\tau)},$$

and satisfies $j(\tau) = (\mathfrak{f}(\tau)^{24} - 16)^3/\mathfrak{f}(\tau)^{24}$.

The coefficients of $\Phi_\ell^{\mathfrak{f}}$ are roughly 72 times smaller.
This means we need 72 times fewer primes.

The polynomial $\Phi_\ell^{\mathfrak{f}}$ is roughly 24 times sparser.
This means we need 24 times fewer interpolation points.

Overall, we get nearly a **1728-fold speedup** using $\Phi_\ell^{\mathfrak{f}}$.

Modular polynomials for $\ell = 11$

Classical:

$$X^{12} + Y^{12} - X^{11}Y^{11} + 8184X^{11}Y^{10} - 28278756X^{11}Y^{9} + 53686822816X^{11}Y^{8}$$
$$- 61058988656490X^{11}Y^{7} + 42570393135641712X^{11}Y^{6} - 17899526272883039048X^{11}Y^{5}$$
$$+ 4297837238774928467520X^{11}Y^{4} - 529134841844639613861795X^{11}Y^{3} + 27209811658056645815522600X^{11}Y^{2}$$
$$- 374642006356701393515817612X^{11}Y + 296470902355240575283200000X^{11}$$

. . . 8 pages omitted . . .

$$+ 392423345094527654908696462408720049099524723370674627089936420642670174061941686739245465600 0 \ldots 000$$

Atkin:

$$X^{12} - X^{11}Y + 744X^{11} + 196680X^{10} + 187X^{9}Y + 21354080X^{9} + 506X^{8}Y + 830467440X^{8}$$
$$- 11440X^{7}Y + 16875327744X^{7} - 57442X^{6}Y + 208564958976X^{6} + 184184X^{5}Y + 1678582287360X^{5}$$
$$+ 1675784X^{4}Y + 9031525113600X^{4} + 1867712X^{3}Y + 32349979904000X^{3} - 8252640X^{2}Y + 74246810880000X^{2}$$
$$- 19849600XY + 98997734400000X + Y^{2} - 8720000Y + 58411072000000$$

Weber:

$$X^{12} + Y^{12} - X^{11}Y^{11} + 11X^{9}Y^{9} - 44X^{7}Y^{7} + 88X^{5}Y^{5} - 88X^{3}Y^{3} + 32XY$$

Computational results

Level records

1. **10009**: Φ_ℓ
2. **20011**: $\Phi_\ell \bmod q$
3. **60013**: $\Phi_\ell^{\mathfrak{f}}$

Speed records

1. **251**:	Φ_ℓ in 28s	$\Phi_\ell \bmod q$ in 4.8s	(vs 688s)
2. **1009**:	Φ_ℓ in 2830s	$\Phi_\ell \bmod q$ in 265s	(vs 107200s)
3. **1009**:	$\Phi_\ell^{\mathfrak{f}}$ in 2.8s		

Effective throughput when computing $\Phi_{1009} \bmod q$ is 100Mb/s.

Single core CPU times (AMD 3.0 GHz), using prime $q \approx 2^{256}$.
Polynomials $\Phi_\ell^{\mathfrak{f}}$ for $\ell < 5000$ available at `http://math.mit.edu/~drew`.

Computing $\phi_\ell(Y)$ with the CRT (naïve approach)

Strategy: lift $j(E)$ from $\mathbb{F}_q$ to $\mathbb{Z}$, compute $\Phi_\ell(X, Y) \bmod p$ and evaluate

$$\phi_\ell(Y) = \Phi_\ell(j(E), Y) \bmod p$$

for sufficiently many primes p. Obtain $\phi_\ell \bmod q$ via the explicit CRT.

Uses $O(\ell^2 \log^{3+\epsilon} p)$ expected time for each p, and $O(\ell^2 \log p)$ space.

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However, “sufficiently many” is now $O(\ell n)$, where $n = \log q$.
Total expected time is $O(\ell^3 n \log^{3+\epsilon} \ell)$, using $O(\ell n + \ell^2 \log \ell)$ space.

This approach is **not very useful**:

- If n is large (e.g. $n \approx \ell$), it takes way too long (quartic in ℓ).
- It n is small (e.g. $n \approx \log \ell$), it doesn’t save any space.

Computing $\phi_\ell(Y)$ with the CRT (Algorithm 1)

Strategy: lift $j(E), j(E)^2, j(E)^3, \ldots, j(E)^{\ell+1}$ from $\mathbb{F}_q$ to $\mathbb{Z}$ and compute

$$\phi_\ell(Y) = \sum c_{ik} j(E)^i Y^k \bmod p$$

for sufficiently many primes p, where $\Phi_\ell = \sum c_{ik} X^i Y^k$.
Obtain $\phi_\ell \bmod q$ via the explicit CRT.

Computing $\phi_\ell(Y)$ with the CRT (Algorithm 1)

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for sufficiently many primes p, where $\Phi_\ell = \sum c_{ik} X^i Y^k$.
Obtain $\phi_\ell \bmod q$ via the explicit CRT.

Now "sufficiently many" is $O(\ell + n)$.

For $n = O(\ell \log \ell)$, uses $O(\ell^3 \log^{3+\epsilon} \ell)$ expected time and $O(\ell^2 \log \ell)$ space (under GRH).
For $n = \Omega(\ell \log \ell)$, the space bound is optimal.

This algorithm can also evaluate the partial derivatives of Φ_ℓ needed to construct normalized equations for $\tilde{E}$ (important for SEA).

Computing $\phi_\ell(Y)$ with the CRT (Algorithm 2)

Strategy: lift $j(E)$ from $\mathbb{F}_q$ to $\mathbb{Z}$ and for sufficiently many primes p compute $\phi_\ell \bmod p$ as follows:

1. For each of $\ell + 2$ j-invariants y_i, compute $z_i = \prod_k (j(E) - j_k)$, where the j_k range over $\ell + 1$ neighbors of y_i in $G_\ell(\mathbb{F}_p)$.
2. Interpolate $\phi_\ell(Y) \in \mathbb{F}_p$ as the unique polynomial of degree $\ell + 1$ for which $\phi_\ell(y_i) = z_i$.

Obtain $\phi_\ell \bmod q$ via the explicit CRT.

Computing $\phi_\ell(Y)$ with the CRT (Algorithm 2)

Strategy: lift $j(E)$ from $\mathbb{F}_q$ to $\mathbb{Z}$ and for sufficiently many primes p compute $\phi_\ell \bmod p$ as follows:

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Obtain $\phi_\ell \bmod q$ via the explicit CRT.

For $n = O(\ell^c)$, uses $O(\ell^3(n+\log\ell)\log^{1+\epsilon}\ell)$ expected time and $O(\ell n + \ell\log\ell)$ space (under GRH).

For $n = O(\log^{2-\epsilon} q)$ the algorithm is faster than computing Φ_ℓ.
For $n = \Omega(\log\ell)$ the space bound is optimal.

If n is $\Omega(\log^2\ell)$ and $O(\ell\log\ell)$, one can use a hybrid approach.
This yields an optimal space bound for all $q > \ell$.

Genus 1 point counting in large characteristic

Algorithms to compute $\#E(\mathbb{F}_q) = q + 1 - t$.

Algorithm	Time	Space
Totally naive	$O(e^{2n+\epsilon})$	$O(n)$
Slightly less naive	$O(e^{n+\epsilon})$	$O(n)$
Baby-step giant-step	$O(e^{n/4+\epsilon})$	$O(e^{n/4+\epsilon})$
Pollard kangaroo	$O(e^{n/4+\epsilon})$	$O(n^2)$
Schoof	$O(n^5 \operatorname{llog} n)$	$O(n^3)$
SEA*	$O(n^4 \log^3 n \operatorname{llog} n)$	$O(n^3 \log n)$
SEA (Φ_ℓ precomputed)	$O(n^4 \operatorname{llog} n)$	$O(n^4)$

*Complexity estimates for SEA-based algorithms are heuristic expected times.

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SEA (Φ_ℓ precomputed)	$O(n^4 \operatorname{llog} n)$	$O(n^4)$
SEA with Algorithm 1	$O(n^4 \log^2 n \operatorname{llog} n)$	$O(n^2 \log n)$
Amortized	$O(n^4 \operatorname{llog} n)$	$O(n^2 \log n)$

*Complexity estimates for SEA-based algorithms are heuristic expected times.

Elliptic curve point-counting record

The number of points on the elliptic curve E defined by

$$y^2 = x^3 + 2718281828x + 3141592653,$$

modulo the 5011 digit prime $q = 16219299585 \cdot 2^{16612} - 1$ is

8323769891444946600619018491391378260069836370604500159309667928183741136740938227669912830997846627009617004020582940190774831705166648378125548174433501
6222360544000538839492022451911485986733819166009550859216525385267852842524240978796544500427958734245859103650693623260065854955676905842760404211102908
0666232135885662070661039670759580341918109430064160840690748363019037103169978894180556726367014400296781983798513562269371401276427209286702254047174078
4700901798590441199208750379215971112344019653309999668029194772178482699210001668960742884085944350942098735441112464897682811881029409157742761498481361
8236133983076302692999418138548552140105778012525989072405641889553339872433242793570967700290860169473820597303300518069505065832587533308670748048008463
6983900427134645786532440716786520228221019906549532681092997885462429828884819162973482390308433067055460432955024817309328704331805327934957448788825063 4
8393787807087351238867988051327037590331790801872453585872437467694874112726738073095037665888862659824866162979710551480066332118269833639587932989704356
2635494364468486039656664278370935750099790919223024134537160958876614320893731637296530257682556027127545666105422232328156220481118882835904832158925287
2815308704965441879416303457576489111718650037380917938646571605607395885788665998491783840002043757298666639706781737384345665795929791423993337711367782
2538016636015241053779745447935639933068437226703067711612870475974728874060256153829424355309461429412863767016010448708725782340275978368434887328902748
7046203327761442798102604298830732855895993246333047147994546492842426742530314565704272126471147496267335652133743455000287920232413723922825839150351274
2950736034773585892234313092778077346572608561779267925193030390180619815277308025700377636113052880114730236382334852026580407537787327013748289451197304
6679428777685534275306203922096387437778405610945156936507440996084119730814303901482626498520813641540064044431078342859539882090926223504237272408488115
4327002269478397116252120617133360022725560655793168849910978673768497963315764527084692590231159741512227876106228667690675220660368352958211682369185130
5917272462661882973355576998865646958429361081809162692181866270338066704102681199813126843679500076625472860409064749186815445274367067586843407055463402
1891358398165724885432541336551115909636456700693477449865260368511045411205847035453060636486512589217930914523201112950463798415869418991750541041378713
1053621887908831837273005465881200616271744801682487747455898185251772280214510451501147795355549876845352299986818351761176510147685763441401085581041504
5320737093505215091386326150432421200754980473235846455348856098791946896114485582546561261445641145852160774738994958659512607430608581213461723630995676
5621860727156856482046150112012015113007122286669299590274282107690933890300810525603558710045399536727403969324912420806389527152955993943311641026894824
2637362685355343705851022198361778056561670358618600362869633742502575881826442008352428041311901722720144145496595474571658798526002326473049911956443052
8793982788520765498812575122123974107524497342781984377646508955766617133740349759685046139983532743454125015169809484605689099695691493817145995186900066
9270969498525993914751206760782244790625122626848753012733495289200641795967184561011226439252930596090031649974277634943933178938510726596259437826466293
3791621325648258956921290293302567147491547700031400327806411025863588895745234111758218534120042661084581341547473844325846515861998968494758420093653389
5253588411116027196086899901474201259197147823729252481394860002922779612554902938159895772157480500747896699702410869127401835778517148930637715216609619
1664750803979956621679571978953552211724552632230710653244433669331067442040140391602456581858747401436772403284080454895800825555079522458369190254711040
6012000284990126494269674951154806364097330589798793851739761556415874133478898662870219506352034178937096525462482561334178354529257317157406885610633216
4105705546182508456321207036745733148635468184175804925273259911659543081743640608001131591890082864513124247013731366137271496104705809743302158675109390
8895745441684195336715770412686321350796787391486224738612016911715739107481092463845433183146882764205897555692474146749724490484592370638492569933497005
0287810480327348970976228332956433891007862170869157707253005290879107235507514013187875766473645717669386184406205549908141812073945776391634588349492676
8779319474713905005440022152143445859931404486738106328557238092332971352015340561574971125269604744349474765249731664806205766695327115242584334178977061
6064557079435236406353029020560141015314103419765029534921177065625577774688408769857858804251711896591035794406728856602939161842221528770720582112364327
7156318567658978483022412314216285445946753013230236619421604149931783961968774559963411288277953692794747738279937358682979368994295124969120288710932706
3284624677436722012981685194580777814009291336645358525962424649443734012222395524 8

Elliptic curve point counting record

Task	Total CPU Time
Compute $\phi_\ell^{\mathfrak{f}}$	32 days
Find a root $\tilde{j}$	995 days
Compute g_ℓ	3 days
Compute $\pi \bmod g_\ell, E$	326 days
Find λ_ℓ	22 days

$\phi_\ell^{\mathfrak{f}}(Y) = \Phi_\ell^{\mathfrak{f}}(j(E), Y)$ was computed for ℓ from 5 to 11681.
Exactly 700 of 1400 were found to be Elkies primes.
Atkin primes were not used.

The largest $\phi_\ell^{\mathfrak{f}}$ was under 20MB in size and took about two hours to compute using 1 core.

Modular polynomial evaluation record

For $\ell = 100019$ and $q = 2^{86243} - 1$ we computed $\phi_\ell^{\mathfrak{f}}(Y) = \Phi_\ell^{\mathfrak{f}}(j(E), Y)$.

This is much larger than one would need to set a 25,000 digit point-counting record.

The size of $\phi_\ell^{\mathfrak{f}}$ is about **1 GB**.

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- The size of $\Phi_\ell^{\mathfrak{f}} \bmod q$ is about **2 TB**.
- The size of $\Phi_\ell \bmod q$ is about **50 TB**.
- The size of Φ_ℓ is more than **10 PB**.