ARITHMETIC EQUIVALENCE

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ABSTRACT. In these lecture notes we give an introduction to the theory of arithmetic equivalence, a notion originally introduced in a number theoretic setting to refer to number fields with the same zeta function. Gassmann established a direct relationship between arithmetic equivalence and a purely group theoretic notion of equivalence that has since been exploited in many other areas, most notably in the spectral theory of Riemannian manifolds by Sunada. We will explicate these results and discuss various applications and generalizations of them.

1. LECTURE 1

Let $K$ be a number field (a finite extension of $\mathbb{Q}$), and let $\mathcal{O}_K$ be its ring of integers (the integral closure of $\mathbb{Z}$ in $K$). The Dedekind zeta function of $K$ is defined by the Dirichlet series

$$\zeta_K(s) := \sum_{I \subseteq \mathcal{O}_K} N(I)^{-s} = \prod_p \left(1 - N(p)^{-s}\right)^{-1}$$

where the sum is over nonzero $\mathcal{O}_K$-ideals, the product is over nonzero prime ideals, and $N(I) := [\mathcal{O}_K : I]$ is the absolute norm. For $K = \mathbb{Q}$ the Dedekind zeta function $\zeta_\mathbb{Q}(s)$ is simply the Riemann zeta function $\zeta(s) := \sum_{n \geq 1} n^{-s}$. As with the Riemann zeta function, the Dirichlet series (and corresponding Euler product) defining the Dedekind zeta function converges absolutely and uniformly to a nonzero holomorphic function on $\operatorname{Re}(s) > 1$, and $\zeta_K(s)$ extends to a meromorphic function on $\mathbb{C}$ and satisfies a functional equation, as shown by Hecke [11].

The Dedekind zeta function encodes many features of the number field $K$: it has a simple pole at $s = 1$ whose residue is intimately related to several invariants of $K$, including its class number, and as with the Riemann zeta function, the zeros of $\zeta_K(s)$ are intimately related to the distribution of prime ideals in $\mathcal{O}_K$. There is also a natural generalization of the Riemann hypothesis, which states that all zeros of $\zeta_K(s)$ that are not on the real line lie on the vertical line $\operatorname{Re}(s) = 1/2$.

It is thus natural to ask the following question: to what extent does $\zeta_K(s)$ determine the field $K$? Number fields with the same Dedekind zeta function are said to be arithmetically equivalent, and our question amounts to asking whether arithmetically equivalent number fields are necessarily isomorphic, and if not, how “non isomorphic” can they be?

Let $K_1$ and $K_2$ be number fields contained in a Galois extension of $\mathbb{Q}$, and let $L = \mathcal{O}_K$ be its ring of integers. The Dedekind zeta function of $K$ is defined by the Dirichlet series

$$\zeta_K(s) := \sum_{I \subseteq \mathcal{O}_K} N(I)^{-s} = \prod_p \left(1 - N(p)^{-s}\right)^{-1}$$

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Let us begin by considering two number fields $K_1$ and $K_2$, and let $L/\mathbb{Q}$ be a finite Galois extension containing both $K_1$ and $K_2$ (the compositum of their Galois closures, for example). Let $G := \operatorname{Gal}(L/\mathbb{Q})$ and let $H_1,H_2 \leq G$ be the subgroups with fixed fields $L^{H_1} = K_1$ and $L^{H_2} = K_2$ given by the Galois correspondence. In 1925 Fritz Gassmann [6] made the remarkable observation that the arithmetic equivalence of $K_1$ and $K_2$ is completely determined by the relationship between $H_1,H_2 \leq G$.

Definition 1.1. Let $H_1,H_2$ be subgroups of a finite group $G$. We say that $H_1$ and $H_2$ are Gassmann equivalent and call $(G,H_1,H_2)$ a Gassmann triple if there is a bijection of set $H_1 \leftrightarrow H_2$ that preserves $G$-conjugacy. Equivalently, for all $g \in G$ we have

$$|(H_1 \cap g^G)| = |(H_2 \cap g^G)|,$$

where $g^G$ denotes the conjugacy class of $G$. 

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If $H_1, H_2 \leq G$ are conjugate then $(G, H_1, H_2)$ is obviously a Gassmann triple; we are only interested in non-trivial Gassmann triples, those in which $H_1$ and $H_2$ are not $G$-conjugate.

**Example 1.2.** Let $p$ be prime and consider the following subgroups of $G := \text{GL}_2(F_p)$:
\[
H_1 := \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in \text{GL}_2(F_p) \right\} \quad \text{and} \quad H_2 := \left\{ \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \in \text{GL}_2(F_p) \right\}
\]
The bijection $\left( \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \right) \leftrightarrow \left( \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \right)$ preserves $G$-conjugacy, so $(G, H_1, H_2)$ is a Gassmann triple. For $p > 2$ the subgroups $H_1$ and $H_2$ are not conjugate in $G$, since $H_1$ fixes a common eigenspace, but $H_2$ does not.

In the previous example the Gassmann equivalent subgroups $H_1$ and $H_2$ are isomorphic, and in fact one can embed $G$ in a larger group $G'$ in which $H_1$ and $H_2$ are $G'$-conjugate. If $(G, H_1, H_2)$ is a Gassmann triple, then $H_1$ and $H_2$ necessarily have the same cardinality (there is a bijection between them), and since the order of an element is determined by its conjugacy class, the groups $H_1$ and $H_2$ must also have the same order statistics, which for any finite group $H$ we define as the integer function
\[
\phi_H : \mathbb{Z} \rightarrow \mathbb{Z}
\]
\[
e \mapsto \# \{ h \in H : |h| = e \};
\]

note that $\phi_H$ depends only on the isomorphism class of $H$ as an abstract group. If $H_1$ and $H_2$ are Gassmann equivalent (as subgroups of any group $G$), then $\phi_{H_1} = \phi_{H_2}$. For any particular $G$ the converse need not hold, but if we work in the category of abstract groups, and give ourselves the freedom to choose $G$, compatibility of order statistics is the only constraint.

**Theorem 1.3.** Let $H_1$ and $H_2$ be finite groups. There exists a finite group $G$ with subgroups $H'_1 \simeq H_1$ and $H'_2 \simeq H_2$ such that $(G, H'_1, H'_2)$ is a Gassmann triple if and only if $H_1$ and $H_2$ have the same order statistics.

**Proof.** As noted above, the forward implication is immediate. For the reverse, assume $\phi_{H_1} = \phi_{H_2}$, let $n := \#H_1 = \#H_2$, let $G := S_n$ be the symmetric group on $n$-letters, and for $i = 1, 2$ let $H'_i \leq G$ be the left regular permutation representation of $H_i$. Each $h \in H'_i$ is a permutation consisting of $n/|h|$ cycles of length $|h|$. Thus $h_1 \in H'_1$ and $h_2 \in H'_2$ are conjugate in $G$ if and only if they have the same order. Any bijection $H_1 \leftrightarrow H_2$ that preserves element orders thus preserves $G$-conjugacy, and since $H'_1$ and $H'_2$ have the same order statistics, such a bijection exists. □

Abelian groups with the same order statistics are necessarily isomorphic, but this is not true in general; the smallest examples of group $H_1 \not\simeq H_2$ with the same order statistics have order 16 (the groups with GAP [5] identifiers $\langle 16, 10 \rangle$ and $\langle 16, 13 \rangle$ are an example). The following example provides an infinite family of non-isomorphic pairs of groups with the same order statistics from which we can construct Gassmann triples via Theorem 1.3 above.

**Example 1.4.** Following [15], let $p$ be an odd prime and let
\[
H_1 := \left( \mathbb{Z}/p\mathbb{Z} \right)^3 \quad \text{and} \quad H_2 := \left\{ \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \in \text{SL}_3(F_p) \right\}.
\]
These groups are non-isomorphic, since $H_1$ is abelian and the Heisenberg group $H_2$ is not, but they both contain $p^3 - 1$ elements of order $p$ and one element of order 1; thus $\phi_{H_1} = \phi_{H_2}$. As in the proof of Theorem 1.3, we can embed $H_1$ and $H_2$ in $G := S_{p^3}$ to obtain a Gassmann triple $(G, H_1, H_2)$ (in fact, we can embed them in a subgroup $G \leq S_{p^3}$ of order $p^6$).

We now state the Gassmann’s main result [6], which can also be found in [3, Ex. 6.4] and [17, Thm. 1].

**Theorem 1.5** (Gassmann, 1925). Number fields $K_1$ and $K_2$ in a finite Galois extension $L$ are arithmetically equivalent if and only if they are the fixed fields of Gassmann equivalent subgroups $H_1, H_2 \leq \text{Gal}(L/\mathbb{Q})$. 

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We will prove this theorem in the next lecture. Note that number fields in Gassmann’s theorem are isomorphic if and only if $H_1$ and $H_2$ are conjugate in $G$. Thus for any finite group $G$ that can be realized as $G = \text{Gal}(L/\mathbb{Q})$ for some Galois extension $L/\mathbb{Q}$, we can construct arithmetically equivalent non-isomorphic number fields $K_1 := L^{H_1}$ and $K_2 := L^{H_2}$ using any non-trivial Gassmann triple $(G, H_1, H_2)$, and every such example arises in this way. Every symmetric group can certainly be realized as the Galois group of a number field, so Example 1.4 already provides many examples, but the degree of the number fields involved may be very large. Below we give another example.

**Example 1.6.** For $G = \text{GL}_2(F_p)$ we can explicitly construct $L/\mathbb{Q}$ with $\text{Gal}(L/\mathbb{Q}) = G$ as in [4]. If $E/\mathbb{Q}$ is an elliptic curve then the field $\mathbb{Q}(E[p])$ generated by the coordinates of its $p$-torsion points in $\overline{\mathbb{Q}}$ is a Galois extension of $\mathbb{Q}$ whose Galois group is isomorphic to a subgroups of

$$\text{Aut}(E[p]) \cong \text{Aut}(\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}) \cong \text{GL}_2(F_p).$$

By Serre’s open image theorem [19], for elliptic curves $E/\mathbb{Q}$ without complex multiplication we will have $\text{Gal}(\mathbb{Q}(E[p])/\mathbb{Q}) \cong \text{GL}_2(F_p)$ for all but finitely many primes $p$, and in fact for almost all $E/\mathbb{Q}$ this will be true for every primes $p$ (by [13], for example). The elliptic curve $y^2 + y = x^3 - x$ with Cremona label $37a1$ is an example, and more than a million others can be found in the L-functions and modular forms database. If we now take $H_1$ and $H_2$ as in Example 1.2, we obtain arithmetically equivalent non-isomorphic number fields $K_1 := \mathbb{Q}(E[p])^{H_1}$ and $K_2 := \mathbb{Q}(E[p])^{H_2}$ of degree $p^2 - 1$. For $p = 3$ we have $p^2 - 1 = 8$, which is nearly the smallest possible (the smallest degree in which one finds arithmetically equivalent non-isomorphic number fields is 7, as shown in [2]).

### 1.1. Hearing the shape of a drum.

One can attach zeta functions to many other mathematical objects. Let us now consider the case of a Riemannian manifold $M$, a smooth manifold equipped with a Riemannian metric $g$. Recall that a smooth manifold (of dimension $n$) is a second countable Hausdorff space equipped with an atlas of charts $\varphi_U : U \to \mathbb{R}^n$ indexed by an open cover $\mathfrak{U}$ such that each chart defines a homeomorphism and the transition maps $\varphi_U \circ \varphi_U^{-1} : \varphi(U \cap V) \to \varphi_V(U \cap V)$ on overlapping charts are (infinitely differentiable) diffeomorphisms. We use $C^\infty(M)$ to denote the $\mathbb{R}$-algebra of smooth functions $f : M \to \mathbb{R}$, those for which $f \circ \varphi_U^{-1}$ is infinitely differentiable for all $U \in \mathfrak{U}$.

The Riemannian metric $g$ is a symmetric positive definite $(0, 2)$-tensor field,\footnote{We follow the (mostly) standard convention that a $(p, q)$-tensor field is $p$-contravariant and $q$-covariant, meaning that its elements can be viewed as smoothly-varying multi-linear functions $T^p_x = \mathbb{R}^q$ equipped with a smooth projection $T^p_x \to \mathbb{R}$ or as smoothly varying elements of $(T_x^p)^q \otimes (T^q_x)^p$, where $T_x$ is the tangent space at $x \in M$ and $T^p_x$ is its dual (the cotangent space).} which we may view as a smoothly varying inner product $\langle \cdot, \cdot \rangle_x$ on the tangent spaces $T_x$; here $T_x$ is the $n$-dimensional $\mathbb{R}$-vector space of $\mathbb{R}$-linear maps $v : C^\infty(M) \to \mathbb{R}$ that satisfy $v(fg) = f(x)v(g) + v(f)g(x)$ for all $f, g \in C^\infty(M)$; these are derivations of $C^\infty(M)$ at $x$ (the tangent space can also be defined using isomorphism classes of curves through $x$). The disjoint union $T(M) := \sqcup_{x \in M} T_x$ is the tangent bundle of $M$; it has a natural structure as a smooth manifold of dimension $2n$ equipped with a smooth projection $T(M) \to M$ whose fibers are tangent spaces. The cotangent bundle $T^*(M)$ is similarly defined using the dual spaces $T^*_x$.

- $\mathcal{J}(M)$ is the $C^\infty(M)$-module of smooth sections of $T(M)$.
  - Elements of $\mathcal{J}(M)$ are $(1, 0)$-tensor fields (vector fields), equivalently, derivations of $C^\infty(M)$.
  - (They define functions $C^\infty(M) \to C^\infty(M)$ by taking directional derivatives).
- $\mathcal{J}^*(M)$ is the $C^\infty(M)$-module of smooth sections of $T^*(M)$.
  - Elements of $\mathcal{J}^*(M)$ are (0, 1)-tensor fields, also known as differential 1-forms.
The metric $g$ can be viewed as a symmetric $C^\infty(M)$-bilinear map $\mathcal{T}(M) \times \mathcal{T}(M) \to C^\infty(M)$. It uniquely determines an isomorphism $\#: \mathcal{T}(M) \to \mathcal{T}(M)$ via $X^\#: (Y \mapsto \langle X, Y \rangle_g)$, which together with its inverse $\#: \mathcal{T}(M) \to \mathcal{T}(M)$ is known as a musical isomorphism.

We also have the Levi-Civita connection $\nabla : \mathcal{T}(M) \times \mathcal{T}(M) \to \mathcal{T}(M)$, the unique torsion-free affine connection compatible with $g$. This means that for all $f \in C^\infty(M)$ and $X, Y, Z \in \mathcal{T}(M)$ we have

- $\nabla(fX, Y) = f \nabla(X, Y)$ and $\nabla(X, fY) = df(X)Y + f \nabla(X, Y)$ (affine connection),
- $\nabla(X, Y) - \nabla(Y, X) = XY - YX$ (torsion free),
- $X(g(Y, Z)) = g(\nabla(X, Y), Z) + g(Y, \nabla(X, Z))$ (compatible with $g$).

Note that while $\nabla$ is $\mathbb{R}$-bilinear, it is $C^\infty(M)$-linear only in the first argument. We can alternatively view the Levi-Civita connection as a $C^\infty(M)$-linear map

$$\mathcal{T}(M) \to (\mathcal{T}(M) \to \mathcal{T}(M))$$

$$X \mapsto \nabla_X := (Y \mapsto \nabla(X, Y)).$$

We define the trace of a $C^\infty(M)$-linear map $\mathcal{T}(M) \to \mathcal{T}(M)$ as the function in $C^\infty(M)$ obtained by taking the trace of the linear map on the tangent space at each point. We now define the $\mathbb{R}$-linear operators

$$\text{grad} : C^\infty(M) \to \mathcal{T}(M) \quad \text{div} : \mathcal{T}(M) \to C^\infty(M)$$

$$f \mapsto df^\#: \quad X \mapsto \text{tr}(\nabla_X),$$

and our main object of interest, the Laplace-Beltrami operator

$$\Delta_M : C^\infty(M) \to C^\infty(M)$$

$$f \mapsto -\text{div} \text{ grad } f \quad \text{(note the sign)}.$$

All of the operators we have defined intrinsically depend on the metric $g$, even though we do not highlight this dependence in our notation. Whenever we refer to a Riemannian manifold $M$ we understand that it is equipped with a Riemannian metric (some authors write $(M, g)$ to emphasize the the metric; we view $g$ is part of the definition of $M$).

**Theorem 1.7.** Let $M$ be a compact connected Riemannian manifold. The eigenspaces of $\nabla_M$ all have finite dimension, and the corresponding eigenvalues form a countable discrete sequence of non-negative real numbers. If we enumerate the eigenvalues with multiplicity as

$$0 \leq \lambda_1 \leq \lambda_2 \leq \cdots$$

then there exists an orthonormal sequence of $C^\infty(M)$ functions $\{f_1, f_2, \ldots\}$ with $\Delta_M f_i = \lambda_i f_i$ that contains a basis for every eigenspace and generates a dense subspace of $L^2(M)$ in the $L^2$-norm topology.

**Proof.** See [1, pp. 54-55] for a sketch of the proof (with references to further details). \qed

The ordered sequence of nonzero eigenvalues of $\Delta_M$ (listed with multiplicity) is the eigenvalue spectrum of $M$, denoted $\lambda(M)$. Riemannian manifolds with the same spectrum are said to be isospectral.

**Definition 1.8.** Let $M$ be a compact connected Riemannian manifold of dimension $n$ with eigenvalue spectrum $\lambda(M) = (\lambda_i)_{i \geq 1}$. The (Minakshisundaram-Pleijel) zeta function of $M$ is defined by the generalized Dirichlet series

$$\zeta_M(s) = \sum_{i \geq 1} \lambda_i^{-s},$$
which converges absolutely and uniformly to a holomorphic function on some right half plane and has a meromorphic continuation to \( \mathbb{C} \) that is holomorphic except for simple poles at integers \( 1, \ldots, n/2 \) if \( n \) is even and half integers \( n/2, n/2 - 1, \ldots \) if \( n \) is odd [16].

**Lemma 1.9.** Let \( M_1 \) and \( M_2 \) be compact connected Riemannian manifolds. Then \( \lambda(M_1) = \lambda(M_2) \) if and only if \( \zeta_{M_1}(s) = \zeta_{M_2}(s) \).

**Proof.** The forward implication is obvious. For the reverse, suppose \( \zeta_{M_1}(s) = \zeta_{M_2}(s) \) but \( \lambda(M_1) \neq \lambda(M_2) \). Without loss of generality we may assume that there is a positive integer \( j \) such that \( \lambda_i(M_1) = \lambda_i(M_2) \) for \( 1 \leq i < j \) and \( \lambda_j(M_1) > \lambda_j(M_2) \). Note that this implies \( \lambda_j(M_1) > \lambda_i(M_2) \) for all \( i \geq j \). Let \( n_j \) be the multiplicity of \( \lambda_j(M_1) \) in \( \lambda(M_1) \) and let \( \sigma \) be the maximum of the abscissa of convergence for the generalized Dirichlet series defining \( \zeta_{M_1}(s) \) and \( \zeta_{M_2}(s) \). We have

\[
\zeta_{M_1}(t) - \zeta_{M_2}(t) \sim n_j \lambda_j(M_1)^{-t}
\]

as \( t \geq \sigma \) tends to infinity along the real line, but this contradicts \( \zeta_{M_1}(s) = \zeta_{M_2}(s) \). \( \square \)

**Example 1.10.** Let \( S^1 \) be the unit circle in \( \mathbb{R}^2 \). Then \( \lambda(S^1) = \{n^2 : n \geq 1 \text{ with multiplicity } 2 \} \) and we have \( \zeta_{S^1}(s) = \sum_{n \geq 1} 2n^{-2s} = 2\zeta(2s) \), where \( \zeta(s) := \sum_{n \geq 1} n^{-s} \) is the Riemann zeta function.

**Definition 1.11.** A (full) lattice \( \Lambda \) in \( \mathbb{R}^n \) is the \( \mathbb{Z} \)-span of a basis for \( \mathbb{R}^n \), the dual lattice is defined by

\[
\Lambda^* := \{ v \in \mathbb{R}^n : \langle v, w \rangle \in \mathbb{Z} \text{ for all } w \in \Lambda \},
\]

where \( \langle \cdot, \cdot \rangle \) is the Euclidean inner product (dot product). We say that \( \Lambda \) is

- **integral** if \( \langle v, w \rangle \in \mathbb{Z} \) for all \( v, w \in \Lambda \) (equivalently, \( \Lambda \subseteq \Lambda^* \));
- **even** if \( \langle v, w \rangle \in 2\mathbb{Z} \) for all \( v, w \in \Lambda \) (implies integral);
- **unimodular** if \( \Lambda \) has covolume \( \mu(\mathbb{R}^n/\Lambda) = 1 \);
- **self dual** if \( \Lambda = \Lambda^* \) (equivalently, \( \Lambda \) is integral and unimodular).

Two lattices in \( \mathbb{R}^n \) are **isomorphic** if they are related by an orthogonal linear transformation.

**Definition 1.12.** Let \( \Lambda \) be a lattice in \( \mathbb{R}^n \). The theta series of \( \Lambda \) is defined by the formal \( q \)-series

\[
\Theta_\Lambda(q) := \sum_{v \in \Lambda} q^{\langle v, v \rangle}/2,
\]

If we substitute \( q = e^{-2\pi t} \) we obtain a holomorphic function on \( \text{Re}(s) > 0 \). The zeta function of \( \Lambda \) is

\[
\zeta_\Lambda(s) := \sum_{v \in \Lambda - \{0\}} \langle v, v \rangle^{-s},
\]

which can be derived from \( \Theta_\Lambda \) by taking a Mellin transform:

\[
\zeta_\Lambda(s) = \pi^s \Gamma(s)^{-1} \int_0^\infty \Theta_\Lambda(e^{-2\pi t}) - 1) t^{s-1} dt.
\]

This formula can be inverted using the inverse Mellin transform, thus \( \zeta_\Lambda \) and \( \Theta_\Lambda \) determine each other.

**Example 1.13.** Let \( M := \mathbb{R}^n/\Lambda \) be the torus defined by a lattice \( \Lambda \) in \( \mathbb{R}^n \) equipped with the flat metric (here "flat" means zero curvature; this is not the metric induced by the standard embedding in \( \mathbb{R}^{n+1} \)). The eigenvalue spectrum of \( M \) is

\[
\lambda(M) = \{4\pi^2 \langle v, v \rangle : \text{nonzero } v \in \Lambda^* \},
\]
with corresponding eigenfunctions are \( \{e^{2\pi i \langle v, x \rangle} : \text{nonzero } v \in \Lambda^* \} \). If \( \Lambda \) is self-dual then
\[
\zeta_M(s) = \sum_{v \in \Lambda - \{0\}} (4\pi^2 \langle v, v \rangle)^{-s} = \zeta_{2\pi\Lambda}(s),
\]
which we note is determined by (and determines) \( \Theta_\Lambda(q) \), since \( \Theta_{2\pi\Lambda}(q) = \Theta_\Lambda(4\pi^2 q) \).

**Example 1.14.** For integers \( n \geq 3 \) the lattice \( D_n \) is defined by
\[
D_n := \{(a_1, \ldots, a_n) \in \mathbb{Z}^n : a_1 + \cdots + a_n \equiv 0 \text{ mod } 2 \};
\]
it is an even lattice but is not self-dual. For even integers \( n \geq 4 \) we define the lattice
\[
D_n^+ := D_n \cup (D_n + h),
\]
where \( h = (\frac{1}{2}, \ldots, \frac{1}{2}) \), which is even and self-dual for \( n \) divisible by 4 (in fact \( D_4^+ \cong \mathbb{Z}^4 \)).

The lattice \( D_8^+ \) is more commonly known as \( E_8 \); up to isomorphism it is the unique even self-dual lattice of dimension 8. In dimension 16 there are two even self-dual lattices (up to isomorphism): \( E_8 \oplus E_8 \) and \( D_{16}^+ \). As first shown by Witt [22], these non-isomorphic lattices have the same theta series. This follows from the fact that the theta series of even self-dual lattices of dimension \( n \) correspond to a modular form of weight \( n/2 \). More precisely for any such lattice, the function
\[
\theta_\Lambda(\tau) := \Theta_\Lambda(e^{2\pi i \tau})
\]
satisfies:

- \( \theta_\Lambda(\tau) \) is holomorphic on the upper half plane \( \text{Im}(\tau) > 0 \);
- \( \theta_\Lambda(\tau + 1) = \theta_\Lambda(\tau) \) and \( \theta_\Lambda(-1/\tau) = \tau^{n/2} \theta_\Lambda(\tau) \);
- \( \theta_\Lambda(\tau) \) remains bounded as \( \text{Im} \tau \to \infty \) in the strip \( 0 \leq \text{Re}(\tau) < 1 \).

This implies that \( \theta_\Lambda(\tau) \) is a modular form of weight \( n/2 \) for the full modular group \( \text{SL}_2(\mathbb{Z}) \). For \( n = 16 \) the space of modular forms of weight \( n/2 = 8 \) for \( \text{SL}_2(\mathbb{Z}) \) has dimension 1, so the \( q \)-series of any two such modular forms differ only by a scalar; the \( q \)-series defining \( \theta_\Lambda(\tau) \) has constant coefficient 1, so there is only one possible theta series for an even self-dual lattice of dimension 16. It follows that the corresponding flat tori have the same zeta function
\[
\zeta_{R^{16}/E_8 \oplus E_8}(s) = \zeta_{R^{16}/D_{16}^+}(s),
\]
as observed by Milnor [17] in 1964.

In 1966 Mark Kac famously asked “Can one hear the shape of a drum?” [14]. Kac was asking whether the eigenvalue spectrum of a compact Riemannian manifold \( M \) in the Euclidean plane determines \( M \) up to isomorphism (for manifolds with boundary one restricts to functions with vanishing normal derivative at the boundary when considering the eigenvalue spectrum). It was already known that isospectral manifolds need not be isomorphic in general, as noted in Example 1.14. More than 25 years passed before Gordon, Webb, and Wolpert negatively answered Kac’s question [8] by extending a general method for constructing non-isomorphic isospectral manifolds that was introduced by Sunada in the mid 1980s. Sunada’s result makes essential use of Gassmann triples.

### 1.2. Riemannian coverings.

We now work in the category of smooth connected manifolds whose morphisms are smooth maps (they induce smooth maps of Euclidean spaces locally).

A **smooth cover** (or **smooth covering**) is a surjective morphism \( \pi : M \to X \) of smooth connected manifolds that is a local diffeomorphism: every point in \( X \) has an open neighborhood \( U \) such that each connected component of \( \pi^{-1}(U) \) is mapped diffeomorphically onto \( U \) by \( \pi \). Smooth covers preserve
A smooth cover \( \pi : M \to X \) is called a universal cover if \( M \) is simply connected. It has the universal property that every smooth cover \( \psi : N \to X \) admits a smooth cover \( \phi : M \to N \) such that \( \pi = \psi \circ \phi \), equivalently, the universal cover is an initial object in the category of covering spaces of \( X \). Connected manifolds are path connected, and this implies that every connected smooth manifold has a universal cover, which is unique up to isomorphism.

The fibers of a smooth cover \( \pi : M \to X \) all have the same cardinality, which we denote \( \deg \pi \) (this follows from our requirement that \( M \) be connected). If this cardinality is finite we say that \( \pi \) is finite. Observe that if \( X \) is compact and \( \pi \) is finite, then \( M \) is also compact.

Given a smooth cover \( \pi : M \to X \), a deck transformation (or covering transformation) is an automorphism of \( M \) that fixes \( \pi \), in other words, a diffeomorphism \( \phi : M \to M \) such that \( \pi \circ \phi = \pi \). The deck transformations of \( \pi \) form a subgroup \( \text{Deck}(\pi) \) of the automorphism group \( \text{Aut}(M) \). If \( \pi \) is the universal cover, then \( \text{Deck}(\pi) \) is isomorphic to the fundamental group \( \pi_1(X) \) (which is independent of the base point because \( M \) is path connected). Every smooth cover \( \pi : M \to X \) induces an embedding \( \pi_1(M) \hookrightarrow \pi_1(X) \) of fundamental groups via post-composition. The action of \( \text{Deck}(\pi) \) on \( M \) is free and properly discontinuous; this means that every point in \( M \) has an open neighborhood whose \( \text{Deck}(\pi) \)-translates are disjoint, which implies that the quotient space \( M / \text{Deck}(\pi) \) is a smooth manifold (in particular, it is Hausdorff), and the projection map \( M \to M / \text{Deck}(\pi) \) is a smooth cover. Similar comments apply to any subgroup of \( \text{Deck}(\pi) \).

The group \( \text{Deck}(\pi) \) acts freely on the fibers of \( \pi \), so \( \# \text{Deck}(\pi) \leq \deg \pi \) (thus if \( \pi \) is finite then so is \( \text{Deck}(\pi) \)). If the action of \( \text{Deck}(\pi) \) on the fibers of \( \pi \) is transitive then it is necessarily simply transitive (equivalently, regular), in which case the following equivalent conditions hold:

- each fiber of \( \pi \) is a \( \text{Deck}(\pi) \)-torsor;
- \( M \) is a homogeneous space for \( \text{Deck}(\pi) \);
- \( M / \text{Deck}(\pi) \cong X \);
- the embedding \( \pi_1(M) \hookrightarrow \pi_1(X) \) induced by \( \pi \) has normal image in \( \pi_1(X) \).

Smooth covers that satisfy these equivalent conditions are said to be normal (or regular, or Galois).

Now suppose \( \pi : M \to X \) is a normal smooth cover. For each subgroup \( H \leq \text{Deck}(\pi) \), the projection map \( M \to M/H \) is a normal smooth cover with Deck transformation group \( H \). Associated to any inclusion of subgroups \( H \leq G \leq \text{Deck}(\pi) \), we have a smooth cover \( \pi_{H,G} : M/H \to M/G \) that sends each \( H \)-orbit in \( M \) to the \( G \)-orbit in which it lies. The projection map \( \pi_{G,H} : M \to M/G \) is equal to the composition \( \pi_{H,G} \circ \pi_H \), where \( \pi_H = \pi_{1,H} \) is the projection map \( M \to M/H \).

When \( G = \text{Deck}(\pi) \) we have \( M/G \cong X \), since \( \pi \) is normal, and we may then view \( \pi_{H,G} : M/H \to X \) as a smooth cover of \( X \) that is an intermediate cover of \( \pi \), meaning that \( \pi = \pi_{H,G} \circ \pi_H \); the smooth cover \( \pi_{H,G} \) is normal if and only if \( H \) is a normal in \( G = \text{Deck}(\pi) \). We thus have a “Galois correspondence” that is directly analogous to a Galois extension of fields.

For each subgroup \( H \leq G \) the fixed field \( L^H \) corresponds to the quotient manifold \( M/H \), and field inclusions \( L^{H_2} \subseteq L^{H_1} \) induced by subgroup inclusions \( H_1 \subseteq H_2 \) correspond to smooth covers \( \pi_{H_1,H_2} : M/H_1 \to M/H_2 \). As with field extensions, we distinguish intermediate covers only up to conjugacy: if \( H_1 \) and \( H_2 \) are conjugate subgroups of \( \text{Deck}(\pi) \) then the quotients \( M/H_1 \) and \( M/H_2 \) are diffeomorphic.
For a Riemannian manifold \( X \), a Riemannian covering of \( X \) is a smooth cover \( \pi: M \to X \) that is also a local isometry of Riemannian manifolds; this means that if \( g \) and \( h \) are the metrics on \( X \) and \( M \) respectively then for every \( q \in M \) and every \( X, Y \in T_q M \) we have

\[
h_q(X, Y) = g_{\pi(q)}(\pi_*X, \pi_*Y).
\]

This condition uniquely determines \( h \), and the metric \( \pi^*g \) defined by the RHS of (1) is a Riemannian metric on \( M \). It follows that any smooth covering \( \pi: M \to X \) of a Riemannian manifold \( X \) can be viewed as a Riemannian covering by equipping \( M \) with the metric \( \pi^*g \), and this is the only possible choice. Note that the metric \( \pi^*g \) is invariant under the action of \( \text{Deck}(\pi) \) (since \( \pi \) is), which ensures that every deck transformation is an isometry. So \( \text{Deck}(M) \subseteq \text{Aut}(M) \), and for every subgroup \( H \leq \text{Deck}(\pi) \) the quotient \( M/H \) is a Riemannian manifold, and the projection \( M \to M/H \) is a Riemannian covering.

### 1.3. Isospectral manifolds.

We can now state the theorem of Sunada, which extends Theorem 1.5 to the setting of Riemannian manifolds in one direction [21, Thm. 1].

**Theorem 1.15** (Sunada, 1985). Let \( \pi: M \to X \) be a finite normal Riemannian covering, \( G := \text{Deck}(\pi) \). For any Gassmann triple \((G,H_1,H_2)\) the Riemannian manifolds \( M/H_1 \) and \( M/H_2 \) are isospectral.

In contrast to the situation with number fields, it may happen that non-conjugate subgroups \( H_1 \) and \( H_2 \) give rise to isomorphic (meaning isometric) Riemannian manifolds. But this cannot happen if \( H_1 \) and \( H_2 \) are non-isomorphic, and (as noted by Sunada), every finite group \( G \) arises as the fundamental group of a compact connected smooth manifold of dimension 4; see [20, §9.4.2]. so we can apply Theorem 1.3 to construct isospectral manifolds using any two non-isomorphic groups \( H_1 \) and \( H_2 \) with the same order statistics, such as \( H_1 = (\mathbb{Z}/p\mathbb{Z})^3 \) and \( H_2 \) the Heisenberg group of order \( p^3 \), as in Example 1.4.

Another family of Riemannian manifolds considered by Sunada are Riemann surfaces, which are complex manifolds of dimension 1, hence smooth (real) manifolds of dimension 2 (not every smooth manifold \( M \) of dimension 2 is a complex manifold, only those that are orientable are). Riemann surfaces are smooth manifolds, but to make them Riemannian manifolds we must equip them with a metric; a standard choice is to give them constant negative curvature, and for Riemann surfaces that arise as quotients of the hyperbolic upper half plane by a Fuchsian group (a discrete subgroup of \( \text{SL}_2(\mathbb{R}) \) which acts on the upper half plane by fractional linear transformations), this is the natural choice. More generally, there is an equivalence of categories between connected compact Riemann surfaces and smooth projective (algebraic) curves over \( \mathbb{C} \), and given any smooth projective curve over a number field, we can obtain a connected compact Riemann surface by embedding our number field in \( \mathbb{C} \).
Prasad and Rajan [18] consider the case of a smooth projective curve $X$ over an arbitrary fields $k$ equipped with an action by a finite group $G$ and prove a result directly analogous to Sunada's theorem: for any Gassmann triple $(G, H, H')$ the curves $X/H_1$ and $X/H_2$ have isogenous Jacobians. When $k$ is a finite field or number field, this amounts to saying that $X/H_1$ and $X/H_2$ have the same zeta function or $L$-function; we will discuss this result further in a later lecture.

1.4. Isospectral graphs. There is a discrete analog of Sunada’s theorem that is applicable to graphs. Let us consider the case of a finite undirected graph $\Gamma = (V, E)$, where $E$ is a finite multiset of unordered pairs of elements of $V$ (multiple edges between vertices are allowed; when we enumerate edges we do so with multiplicity). Let $R(V)$ and $R(E)$ denote the $R$-vector spaces of real valued functions on $V$ and the set of edges in $E$ (without multiplicity). The coboundary operator $\nabla$ is the linear map $\nabla : R(V) \to R(E)$ defined by

$$(\nabla f)(\{v, w\}) := f(v) - f(w),$$

which can be viewed as a discrete analog of the gradient. Its transpose $\nabla^\top : R(E) \to R(V)$ is defined by

$$(\nabla^\top g)(v) := \sum_{\{v, w\} \in E} g(\{v, w\}),$$

and can viewed as a discrete analog of divergence. The discrete Laplace operator $\Delta_\Gamma : R(V) \to R(V)$ is the composition $\Delta_\Gamma := \nabla^\top \nabla$, acting on $f \in R(V)$ via

$$(\Delta_\Gamma f)(x) := \sum_{\{v, w\} \in E} (f(v) - f(w));$$

note that the sum enumerates edges with multiplicity, and that self-loops (edges of the form $\{x, x\}$) make no contribution. The eigenvalues of $\Delta_\Gamma$ are the eigenvalues of the matrix $D - A$, where $D$ is the degree matrix of $\Gamma$, and $A$ is its adjacency matrix. If we index the vertices as $V = \{v_1, \ldots, v_n\}$, then $D$ is the diagonal matrix with $D_{ii}$ equal to the number of edges incident to $v_i$ (with multiplicity) and $A_{ij}$ is the symmetric matrix with $A_{ij}$ equal to the multiplicity of $\{v_i, v_j\}$ in $E$ (zero if $\{v_i, v_j\} \notin E$).

The eigenvalues of $\Delta_\Gamma$ are real and nonnegative; the eigenvalue $0$ occurs with multiplicity equal to the number of connected components of $\Gamma$. If we let $0 < \lambda_1 \leq \ldots \leq \lambda_r$ denote the nonzero eigenvalues of $\Delta_\Gamma$ (with multiplicity), the spectral zeta function of $\Gamma$ is defined by

$$\zeta_\Gamma(s) := \sum_{1 \leq i \leq r} \lambda_i^{-s}.$$ 

These definitions can be extended to countably infinite graphs of bounded degree (including lattices) by replacing $R(V)$ with a suitable Hilbert space of functions (typically a reproducing kernel Hilbert space).

An automorphism of $\Gamma$ is a permutation $\varphi : V \to V$ that fixes $E$; this means that $E = \{\varphi(x), \varphi(y)\} : \{x, y\} \in E$. The set of automorphisms of $\Gamma$ form a subgroup $\text{Aut}(\Gamma)$ of the group of permutations of $V$. For any subgroup $H \leq \text{Aut}(\Gamma)$, the quotient graph $\Gamma/H$ is the graph whose vertex set consists of $H$-orbits $[x]$ of $x \in V$ with the multiset of edges $\{\{[x], [y]\} : \{x, y\} \in E\}$.

The following analog of Sunada’s theorem is due to Halbeisen and Hungerbühler [9].

**Theorem 1.16.** Let $\Gamma$ be a finite connected graph and let $H_1$ and $H_2$ be subgroups of $G := \text{Aut}(\Gamma)$ whose non-trivial elements have no fixed points. If $(G, H_1, H_2)$ is a Gassmann triple then $\zeta_{\Gamma/H_1}(s) = \zeta_{\Gamma/H_2}(s)$.

As with Sunada’s theorem, non-conjugate $H_1$ and $H_2$ may yield isomorphic $\Gamma/H_1$ and $\Gamma/H_2$. The assumption that the non-trivial elements of $H_1$ and $H_2$ contain no fixed points is stronger than necessary, but it cannot be completely removed. If $\Gamma$ is a simple graph, the quotient graphs $\Gamma/H_i$ need not be simple, they may contain self-loops and multiple edges. We can remove self-loops, since these do not impact
the spectral zeta function, and as explained in [9], one can modify $\Gamma$ in a way that does not change its spectral zeta function or its automorphism group so that the quotient graphs $\Gamma/H_i$ will be simple and still have the same spectral zeta functions.

REFERENCES


