Supersingular Curves with Small Non-integer Endomorphisms

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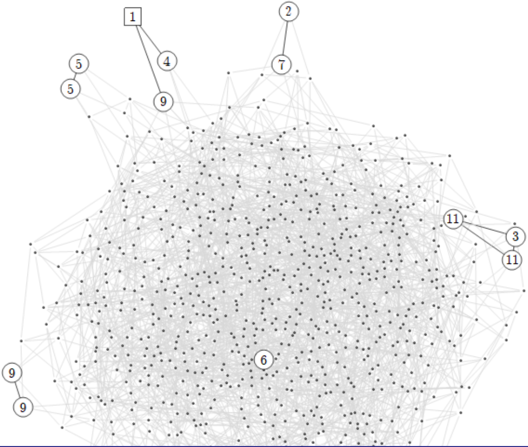
Algorithmic Number Theory Symposium, June 2020

Preview



Main Goal

Describe a manageable subclass of supersingular curves and analyze its structure.

Outline

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Elliptic Curves

Throughout:

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- $p \geq 5$ is a prime,
- F is a finite field of characteristic p,
- E and E' are elliptic curves defined over F.

Isogenies

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Example

If $E : y^2 = x^3 + x$ and $E' : y^2 = x^3 - 4x$, then

$$(x, y) \mapsto \left(\frac{y^2}{x^2}, \frac{y(x^2-1)}{x^2}\right)$$

is an isogeny of degree 2 from E to E', with kernel $\{(0,0), O\}$.

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If $E : y^2 = x^3 + x$, then

$$(x, y) \mapsto (-x, iy)$$

is a non-integer endomorphism of degree 1.

Endomorphism Ring

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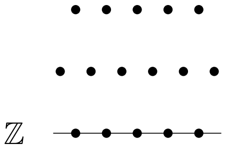

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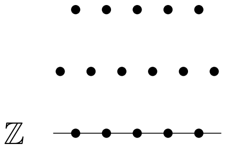

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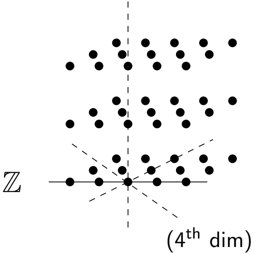

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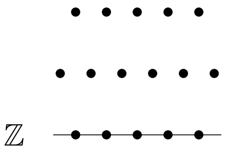

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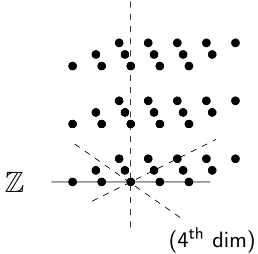

In both cases, degree = norm.

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ℓ-isogeny graphs

Let $\ell \neq p$ be a prime, and define a graph as follows:

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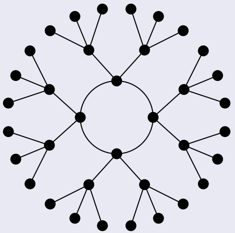

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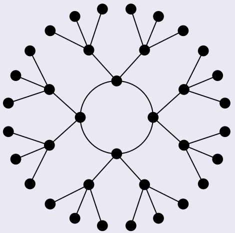

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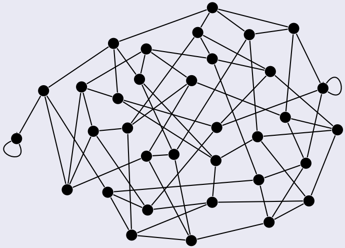

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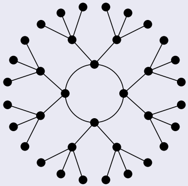

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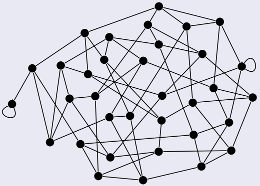

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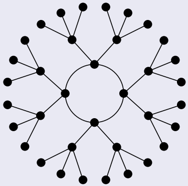

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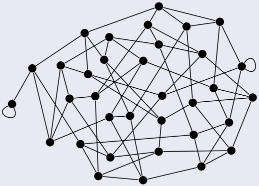

- Each ordinary component has the structure of a volcano.[1]
- There is a unique supersingular component, and it has the structure of a Ramanjuan graph[2] (implies that random walks converge rapidly to the uniform distribution).

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[2] Pizer, A.K. *Ramanujan Graphs and Hecke Operators*, Bulletin of the AMS, Volume 23, Number 1, July 1990.

Supersingular Isogeny Graphs in Cryptography

Hard Problem:

Given two random supersingular curves E and E', find an isogeny $E \to E'$.

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- And more!

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Bröker's algorithm[6] finds one supersingular curve:

- Find an elliptic curve with complex multiplication, defined over a number field K.
- Reduce modulo a prime of K dividing p.
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Then take a random walk in an ℓ-isogeny graph to obtain a random supersingular curve.

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Finding hard supersingular curves

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Motivation: In some cryptographic applications,[7] knowing $\mathrm{End}(E)$ allows for the creation of backdoors. A hard curve would eliminate the need for trusted setup.

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Open Problem

Find an explicit hard supersingular elliptic curve.

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Consider:

- p: A "cryptographic" prime (e.g. $p \sim 2^{200}$)
- M: A "reasonable" parameter (e.g. $M \sim 2^{10}$)

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Definition

An elliptic curve E over a finite field of characteristic p is M-**small** if there exists $\alpha \in \mathrm{End}(E) - \mathbb{Z}$ with $\deg \alpha \leq M$.

Visualizing M-small endomorphism rings

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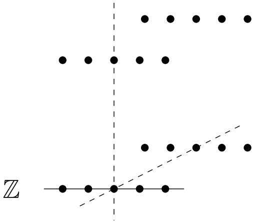

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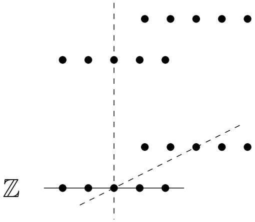

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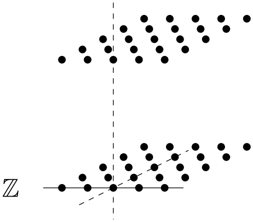

Smallest degree of non-integer endomorphism $\leq M$

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- The set of M-small supersingular curves forms "clusters" indexed by fundamental discriminants.

"Clustering" Theorem

Theorem 1.3

Suppose $p \gg M$. The set of M-small supersingular curves partitions into sets T_D, for fundamental discriminants $-4M \leq D < 0$ with $\left(\frac{D}{p}\right) = -1$.

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[a] One may need to replace E' with its Frobenius conjugate $E'^{(p)}$.

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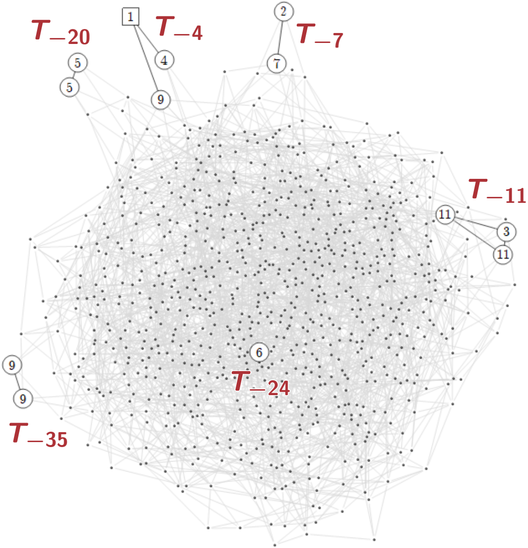

Figure: Supersingular curves in characteristic $p = 20011$ (modulo conjugation on $\mathbb{F}_{p^2}$). Edges: isogenies of prime degree at most $\frac{4}{\pi}\sqrt{12} \approx 4.4$. 12-small curves labelled with smallest degree of a non-integer endomorphism.

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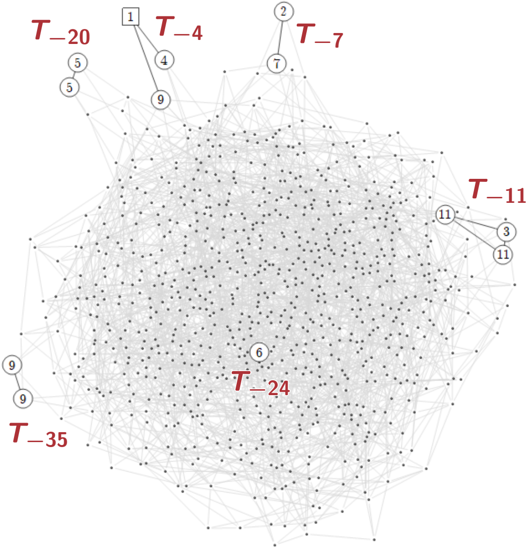

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Despite their distance, we can compute (large-degree) isogenies between the clusters!

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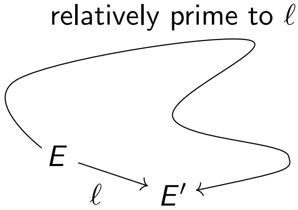
In other words,



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Thank you for listening!

Questions/Comments?

jonlove@stanford.edu dabo@cs.stanford.edu