# Principal polarizations and Shimura data for families of cyclic covers of the projective line

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# The synergy between curves and abelian varieties

Let's say you want to study elliptic curves
or more generally curves of genus $g$ where $1 \leq g \leq 3$.

**Very helpful fact:**

If $1 \leq g \leq 3$, then almost every principally polarized abelian variety $A$ of dimension $g$ is the Jacobian of a smooth curve of genus $g$. (*1) (*2)

There is more information about abelian varieties than about curves. For many years, I leveraged this fact to prove things about curves, even about curves of arbitrarily large genus.

**This project is an opportunity for the curves to give back.**

**Technical notes:**
(*1) If not, then $A$ is the Jacobian of a curve of compact type.
(*2) Geometric statement: the image of the Torelli morphism $\mathcal{M}_g \to \mathcal{A}_g$ is open and dense if $1 \leq g \leq 3$.

# Bird's eye view

We study curves that are cyclic covers of the projective line $\mathbb{P}^1$.
Their invariants: degree $m$, number of branch points $N$, genus $g$, etc.
**Hurwitz spaces** parametrize (families of) curves with same invariants.

The Jacobians of the curves are p.p. abelian varieties. Their invariants: dimension $g$, endomorphisms, signature. **Shimura varieties** parametrize (families of) abelian varieties with same invariants.

## Main result:

Under a condition on the class number of $\mathbb{Q}(\zeta_m)$, for an arbitrary $N$, we determine the Hermitian form and integral Shimura datum of the component of the Shimura variety containing the Torelli locus.

**General strategy:** identify point in family whose Jacobian has CM; explicitly compute this Jacobian, as $\mathbb{C}^g/\Lambda$;
write down Hermitian form coming from its polarization as a Jacobian.

# If we stick to the slides, this is the order of topics:

**Key ingredients:** boundaries of Hurwitz spaces, narrow class numbers of cyclotomic fields, an **algorithm** of Van Wamelen about principal polarizations for abelian varieties with complex multiplication.

1. (Moduli spaces of) cyclic covers of $\mathbb{P}^1$.
2. Algebraic Number Theory:
   complex multiplication,
   principal polarizations,
   independent units,
   narrow class groups.
3. Main result
4. Examples
5. Simple types and future directions

# Introduction - cyclic covers of the projective line

Degree $m$ cyclic cover $X \to \mathbb{P}^1$ with $N$ branch points not ramified at $\infty$ has affine equation

$$X : y^m = \prod_{i=1}^{N} (x - b_i)^{a_i}.$$

(Note - the singularities are no big deal - there is a unique smooth projective curve which has this affine equation away from $\{b_i\}$.)

Inertia type $(a_1, \ldots, a_N)$ with $\sum a_i \equiv 0 \bmod m$ and $\gcd(a_1, \ldots, a_N, m) = 1$.

Fix **monodromy datum** $\gamma = (m, N, a)$. The genus of $X$ is:

$$g = g(\gamma) = 1 + \frac{1}{2}\Big((N-2)m - \sum_{i=1}^{N} \text{gcd}(a(i), m)\Big).$$

# The signature of the Jacobian

Given the curve $X : y^m = \prod_{i=1}^{N}(x - b_i)^{a_i}$.

The Jacobian $\mathrm{Jac}(X)$ is a principally polarized abelian variety of dimension $g$ with an action by $\mathbb{Z}[\mu_m]$.

Decompose $H^0(X, \Omega^1) \simeq \oplus_{i=1}^{m-1} L_i$,
where $L_i$ is the eigenspace given by $\omega \in L_i$ iff $\zeta \cdot \omega = \zeta^i \omega$.

The **signature type** is $f = (f_1, \ldots, f_{m-1})$, where $f_i = \dim(L_i)$.
Note $g = \sum_{i=1}^{m-1} f_i$.

Kani: formula for each $f_i$ in terms of $a$.

$$f_i = -1 + \sum_{j=1}^{N} \langle \frac{-ia_j}{m} \rangle.$$

The data of $a$ is equivalent to the data of $f$, up to equivalence.

# An example that will show up throughout the talk

Degree $m = 5$. Number of branch points $N = 4$.

Choose inertia type $a = (1, 2, 3, 4)$. Then genus $g = 4$.

$X : y^5 = (x - b_1)(x - b_2)^2(x - b_3)^3(x - b_4)^4$.

Easier equation: $X : y^5 = x(x - 1)^2(x - t)^3$.

Signature $f = (1, 1, 1, 1)$ since $f_i = -1 + \langle \frac{-i}{5} \rangle + \langle \frac{-2i}{5} \rangle + \langle \frac{-3i}{5} \rangle + \langle \frac{-4i}{5} \rangle$.

In $\mathcal{M}_4$, the family of these curves has dimension 1.

In $\mathcal{A}_4$, the family of p.p. abelian varieties with an action of $\mathbb{Z}[\zeta_5]$ with signature $f$ has dimension 2.

# Overview - moduli spaces. Let $\gamma = (m, N, a)$.

**Hurwitz space:** $Z_\gamma$ is moduli space of $\mu_m$-covers with type $\gamma$.

$Z_\gamma$ is irreducible. There is a morphism $Z_\gamma \to \mathcal{M}_g$.

Also, $\gamma$ determines a PEL-type **Shimura variety** $\mathrm{Sh}(m, f)$.
Its points represent p.p. abelian varieties with an action of $\mathbb{Z}[\mu_m]$ of signature $f$. It can have many components.

Let $S_\gamma$ be the component containing the image of $Z_\gamma$ under the Torelli morphism $T$. There is a morphism $S_\gamma \to \mathcal{A}_g$.

**Goal**

Determine the integral Shimura datum for $S_\gamma$.

# What is the integral Shimura datum?

Integral Shimura datum for $\mathcal{A}_g$: (i) vector space $V = \mathbb{Q}^{2g}$;
(ii) lattice $\Lambda = \mathbb{Z}^{2g}$;
(iii) standard symplectic form $\Psi : V \times V \to \mathbb{Q}$, integral on $\Lambda$.

Group $G = \mathrm{GSp}(V, \Psi)$ is the group of symplectic similitudes over $\mathbb{Q}$.
Homomorphisms $h : \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \to G_{\mathbb{R}}$ that define Hodge structures.

Add action by $\mathbb{Q}[\mu_m]$: group $H = \mathrm{GL}_{\mathbb{Q}[\mu_m]}(V) \cap G$.

The data of signature $f$ is equivalent to data of orbit of $\{h\}$ under $H(\mathbb{R})$.

**Goal:** *integral Shimura datum for $S_\gamma$*
(i) express $V$ as a vector space over $F = \mathbb{Q}(\zeta_m)$;
(ii) find a $\mathcal{O}_F$-lattice $\Lambda \subset V$; and
(iii) explicitly find Hermitian form $\langle \cdot, \cdot \rangle$ on $V$, taking integral values on $\Lambda$.

# Digression on dimension

The dimension of the Hurwitz family grows linearly with $g$.
$\dim(Z_\gamma) = N - 3$

The dimension of the Shimura variety grows quadratically with $g$.
For $m$ odd, $\dim(S_\gamma) = \sum_{i=1}^{(m-1)/2} f_i f_{m-i}$.

The monodromy type $\gamma$ (and $Z_\gamma$ and $S_\gamma$) are *special* if $\dim(Z_\gamma) = \dim(S_\gamma)$.

If special, then $T(Z_\gamma)$ is open and dense in $S_\gamma$; its points represent Jacobians of smooth curves.

## Remark

Our method does not depend on whether $\gamma$ is special or not.
It applies regardless of the codimension of $T(Z_\gamma)$ in $S_\gamma$.

**Moonen:** Up to equivalence, there are exactly 20 $\gamma$ which are special.
The example $\gamma = (5, 4, (1, 2, 3, 4))$ is not special.
$M[11]$ given by $\gamma = (5, 4, (1, 3, 3, 3))$ has $f = (1, 2, 0, 1)$ and is special.

# Algebraic number theory - Complex multiplication CM

Let $L$ be a CM-field of degree $2n$ over $\mathbb{Q}$.

Let $L_0$ be its maximal totally real subfield; it has degree $n$ over $\mathbb{Q}$.

A CM-type of $L$ is an ordered set $\Phi$ of distinct embeddings $\phi_i : L \hookrightarrow \mathbb{C}$, for $1 \leq i \leq n$, no two of which are complex conjugate.

A CM-type of $L$ is *simple* if it is not induced from the CM-type of a proper CM-subfield of $L$.

Let $A$ be a complex torus such that $L \subset \mathrm{End}(A) \otimes \mathbb{Q}$.

We say that $A$ is of type $(L, \Phi)$ if the complex representation of $\mathrm{End}(A) \otimes \mathbb{Q}$ on the tangent space of $A$ is isomorphic to $\sum_{\phi \in \Phi} \phi$.

This implies that $\dim(A) = n$.

If, in addition, $\mathrm{End}(A) \simeq O_L$, we say $A$ has type $(O_L, \Phi)$.

# The example that everything depends on

Let $m$ be prime and $L = \mathbb{Q}(\zeta_m)$.

Let $\sigma_j : L \to \mathbb{C}$ be the embedding taking $\zeta_m \mapsto (e^{2\pi i/m})^j$.

## The Jacobian of a 3-branch point cover

Let $J = \mathrm{Jac}(C)$ where $C \to \mathbb{P}^1$ is a $\mu_m$-cover with $N = 3$ branch points.

Then $J$ has type $(\mathcal{O}_L, \Phi)$ where $\sigma_j \in \Phi$ iff $f_j > 0$.

E.g. let $m = 5$ and $N = 3$ and $a = (1, 2, 2)$.

Let $C \to \mathbb{P}^1$ have affine equation $y^5 = x(x-1)^2$.

This has $f = (1, 0, 1, 0)$ and $\Phi = \{\sigma_1, \sigma_3\}$;

# CM-theory as found in Lang

Let $L$ be a CM-field and let $\Phi$ be a CM-type of $L$.

If $(L, \Phi)$ is simple, then the class group of $L$ is in bijection with the set of isomorphism classes of complex tori $A$ of type $(\mathcal{O}_L, \Phi)$.

More generally:

1. If $\mathfrak{a}$ is a lattice in $L$, then $\mathbb{C}^n/\Phi(\mathfrak{a})$ is a complex torus of type $(L, \Phi)$.
2. If $A$ is a complex torus of type $(L, \Phi)$, then there exists a lattice $\mathfrak{a}$ of $L$ such that $A \simeq \mathbb{C}^n/\Phi(\mathfrak{a})$.
3. If $\Phi$ is simple and $\mathfrak{a}$ is a fractional ideal of $L$, then $\mathrm{End}(\mathbb{C}^n/\Phi(\mathfrak{a})) \simeq \mathcal{O}_L$, so $\mathbb{C}^n/\Phi(\mathfrak{a})$ has type $(\mathcal{O}_L, \Phi)$.
4. If $\Phi$ is simple and $\mathfrak{a}, \mathfrak{b}$ are fractional ideals of $L$, then $\mathbb{C}^n/\Phi(\mathfrak{a}) \simeq \mathbb{C}^n/\Phi(\mathfrak{b})$ iff $\mathfrak{a}$ and $\mathfrak{b}$ are in the same ideal class.

# Defining a principal polarization

Let $L$ be a CM-field with maximal totally real subfield $L_0$.

Let $\xi \in L$ be such that $L = L_0(\xi)$, $\xi^2 \in L_0$, and $D_{L/\mathbb{Q}} \cdot \mathfrak{a}\bar{\mathfrak{a}} = \langle \xi^{-1} \rangle$, for some fractional ideal $\mathfrak{a}$ of $L$;

The Riemann form $\mathbb{E} : L \times L \to \mathbb{C}$, is given, for $x, y \in L$, by

$$\mathbb{E}(\Phi(x), \Phi(y)) = \mathrm{tr}_{L/\mathbb{Q}}(\xi x \bar{y}), \text{ for } x, y \in L$$
$$\mathbb{E}(z, w) = \sum_{i=1}^{n} \phi_i(\xi)(\bar{z}_i w_i - z_i \bar{w}_i), \text{ for } z, w \in \mathbb{C}^n.$$

## Theorem: Van Wamelen

If $\mathrm{Im}(\phi(\xi)) > 0$ for $\phi \in \Phi$ then $\mathbb{E}$ defines a principal polarization of type $(L, \Phi)$ on $\mathbb{C}^n / \Phi(\mathfrak{a})$.

Furthermore, if $(L, \Phi)$ is a simple CM-type, then all principal polarizations of type $(L, \Phi)$ on $\mathbb{C}^n / \Phi(\mathfrak{a})$ are given by such a $\xi$.

# Van Wamelen's algorithm

## Algorithm

1. Find an ideal $\mathfrak{a}$ of $L$ which represents each ideal class $\mathfrak{A}$ such that $\mathfrak{A}\bar{\mathfrak{A}}$ is the ideal class of the codifferent $D_{L/\mathbb{Q}}^{-1}$ of $L$ over $\mathbb{Q}$.
2. For each $\mathfrak{a}$ as in (1), find $b \in \mathcal{O}_{L_0}$ satisfying $\langle b \rangle = D_{L/\mathbb{Q}} \cdot \mathfrak{a}\bar{\mathfrak{a}}$.
3. If there exists a unit $u \in U_L$ such that $ub = -\overline{ub}$, set $\xi_0 = (ub)^{-1}$.
4. Find representatives $u^+ \in U_{L_0}$ of the cosets of $N_{L/L_0}(U_L)$ in $U_{L_0}$. For each $u^+$, if $\mathrm{Im}(\phi(u^+\xi_0)) > 0$ for each $\phi \in \Phi$, set $\xi = u^+\xi_0$.

Then $\xi$ defines a principal polarization of type $\Phi$ on $\mathbb{C}^n/\Phi(\mathfrak{a})$.
If $(L, \Phi)$ is simple, then this algorithm finds all isomorphism classes of p.p.'s abelian varieties $A$ of type $(\mathcal{O}_L, \Phi)$.

# A special case

## Corollary

Let $\Phi$ be a CM-type of $L$. An element $\beta \in \mathcal{O}_L$ defines a principal polarization on $A_\Phi = \mathbb{C}^n/\Phi(\mathcal{O}_L)$ of CM-type $(L, \Phi)$ if and only if

1. $\beta$ generates the codifferent $D_{L/\mathbb{Q}}^{-1}$;
2. $\beta = -\overline{\beta}$;
3. $\mathrm{Im}(\phi(\beta)) > 0$, for each $\phi \in \Phi$.

Two elements $\beta, \beta'$ satisfying the above conditions yield isomorphic principally polarized abelian varieties if and only if there exists a unit $u \in U_L$ such that $\beta = u\bar{u}\beta'$.

Furthermore, if the CM-type $\Phi$ is simple, then all principal polarizations of $A_\Phi$ of CM-type $(\mathcal{O}_L, \Phi)$ arise this way.

# Example: computation of principal polarization

Let $F = \mathbb{Q}(\zeta_m)$ and $F_0 = \mathbb{Q}(\zeta_m + \zeta_m^{-1})$.
We can explicitly compute $\beta_0 \in \mathcal{O}_F$ satisfying conditions (1) and (2).

**Lemma: let *m* be an odd prime**

The element $\beta_0$ below generates the codifferent $D_{F/\mathbb{Q}}^{-1}$ and $\beta_0 = -\overline{\beta}_0$:

$$\beta_0 = m/(\zeta_m^{(m+1)/2} - \zeta_m^{(m-1)/2}).$$

Reason: $\beta_0$ is on the imaginary axis so (2) true.

For $m$ an odd prime, the $m$th cyc. poly. is $c_m(x) = (x^m - 1)/(x-1)$.

The codifferent $D_{F/\mathbb{Q}}^{-1}$ is generated by $\langle c_m'(\zeta_m) \rangle$,

Compute $c_m'(\zeta_m) = m\zeta_m^{-1}/(\zeta_m - 1)$ and $\beta_0$ is an associate of this.

# Example: computation of principal polarization page 2

We still need to deal with condition (3): $\mathrm{Im}(\phi(\beta)) > 0$, for each $\phi \in \Phi$.

Strategy: observe the signs of $\mathrm{Im}(\phi(\beta_0))$.
Find a totally real unit $u$ s.t. $\beta = u\beta_0$ has the right signs.

Is this always possible?

## Example

Let $m = 5$. Let $F = \mathbb{Q}(\zeta_5)$. Let $F_0 = \mathbb{Q}(\zeta_5)$. Then $\beta_0 = 5/(\zeta_5^3 - \zeta_5^2)$.
Embeddings: $\sigma_j : F \to \mathbb{C}$, taking $\zeta_5 \mapsto \zeta_5^j$.

If $\Phi_1 = \{\sigma_1, \sigma_3\}$, set $u_1 = -1$ so that $\beta_1 = -\frac{5}{\zeta_5^3 - \zeta_5^2}$.

If $\Phi_2 = \{\sigma_1, \sigma_2\}$, set $u_2 = -\frac{\zeta_5^3 - \zeta_5^2}{\zeta_5 - \zeta_5^4}$ so that $\beta_2 = -\frac{5}{\zeta_5 - \zeta_5^4}$.

# Units of independent signs

Let $L$ be a CM-field, with real subfield $L_0$ and $n = \deg(L_0/\mathbb{Q})$.

Fix an ordering $\tau_1, \ldots, \tau_n$ of the real embeddings of $L_0$.

Let $U_{L_0}$ be the units of $L_0$ and let $u \in U_{L_0}$.

Define $\rho_{L_0} : U_{L_0} \to \{\pm 1\}^n$ by $\rho_{L_0}(u) = (\tau_i(u)/|\tau_i(u)|)_{1 \leq i \leq n}$.

Say $L_0$ has *units of independent signs* if $\rho_{L_0}$ is surjective.

Also $L_0$ has *units of almost independent signs* if $|\mathrm{coker}(\rho_{L_0})| = 2$.

# Connection with the narrow class group

Let $L$ be a CM-field, with real subfield $L_0$ and $n = \deg(L_0/\mathbb{Q})$.

Fix an ordering $\tau_1, \ldots, \tau_n$ of the real embeddings of $L_0$.

We say that $\alpha \in L_0$ is *totally positive* if $\tau_i(\alpha) > 0$ for $1 \leq i \leq n$.

The narrow class group is $\mathrm{cl}_{L_0^+} = I_{L_0}/P_{L_0}^+$, where $P_{L_0}^+$ is the group of principal ideals generated by a totally positive element.

There is a surjection $\nu_{L_0} : \mathrm{cl}_{L_0}^+ \to \mathrm{cl}_{L_0}$.

Then $L_0$ has units of independent signs iff $\nu_{L_0}$ is an isomorphism.

**Fact:** The narrow class number $h_{L_0}^+$ is odd (resp. 1) iff the class number $h_{L_0}$ is odd (resp. 1) and $L_0$ has units with independent signs.

Also $L_0$ has units of almost independent signs iff $|\ker(\nu_{L_0})| = 2$.

# Principal polarizations and independent signs

Let $L/L_0$ be a CM-extension.
If $(L, \Phi)$ is a CM-type, there is a p.p. CM-abelian variety of that type.

## Proposition

Suppose $L_0$ has units of independent signs.

1. If $(L, \Phi)$ is simple, then there exists a p.p. CM-abelian variety $A$ of type $(\mathcal{O}_L, \Phi)$. Also $A$ has at most one principal polarization, up to isomorphism.
2. Suppose, in addition, that $L$ has class number 1. Then for any simple CM-type $(L, \Phi)$, there exists a unique p.p. CM-abelian variety of type $(\mathcal{O}_L, \Phi)$, up to isomorphism.

# Proof

There exists a fractional ideal $\mathfrak{a}$ in $L$ and $\beta_0 \in \mathcal{O}_L$ satisfying
(1) $\beta_0$ generates $D_{L/\mathbb{Q}}\mathfrak{a}\bar{\mathfrak{a}}$ and (2) $\beta_0 = -\bar{\beta}_0$.
If $L_0$ has units of independent signs, then there exists $u_0 \in U_{L_0}$ s.t. $\beta = u_0\beta_0$ satisfies (3) $\mathrm{Im}(\phi(\beta)) > 0$, for each $\phi \in \Phi$.
Then $\beta \in \mathcal{O}_L$ defines a p.p. of type $(L, \Phi)$ on $A = \mathbb{C}^n/\Phi(\mathfrak{a})$.

If $(L, \Phi)$ is simple, all p.p.'s of type $(L, \Phi)$ on $\mathbb{C}^n/\Phi(\mathfrak{a})$ arise from some $\beta \in \mathcal{O}_L$ satisfying (1)–(3). If $\beta, \beta' \in \mathcal{O}_L$ satisfy (1)–(3), then $\beta' = u^+\beta$ for some totally positive unit $u^+ \in U_{L_0}^+$.
Since $L_0$ has units of independent signs, $U_{L_0}^+ = N_{L/L_0}(U_L)$. So $u^+ = N(u)$ for some unit $u \in U_L$, thus $\beta, \beta'$ define isomorphic p.p.'s on $A$.

If $L$ has class number 1, then any CM-abelian variety of type $(L, \Phi)$ is isomorphic to $A_\Phi = \mathbb{C}^n/\Phi(\mathcal{O}_L)$.

# Class groups of cyclotomic fields

For example, let $m$ be a positive odd integer.
Take $L$ to be the cyclotomic field $F = \mathbb{Q}(\zeta_m)$, which is a CM-field.
Its real subfield is $F_0 = \mathbb{Q}(\zeta_m + \zeta_m^{-1})$, with degree $n = \phi(m)/2$ over $\mathbb{Q}$.

By Lang: if $(F, \Phi)$ is a simple CM-type, then there is a bijection between isomorphism classes of complex tori of type $(\mathcal{O}_F, \Phi)$ and elements of the class group $\mathrm{cl}_F$ of $F$.

## Restriction

Assume that $F$ has class number 1 (i.e., $\mathcal{O}_F$ has unique factorization).

This also implies that $F_0$ has class number 1.

Finite list of odd $m$:
Primes: $3, 5, 7, 9, 11, 13, 17, 19, 25, 27$
Prime powers: $9, 25, 27$
Not prime powers: $15, 21, 33, 35, 45$

# The narrow class number of cyclotomic fields

Let $t$ be the number of finite primes ramified in $L/L_0$.

**Found in Conner/Hurrelbrink**

Suppose $L_0$ is a totally real field, and $L/L_0$ is a CM-extension.
Then $h_L$ is odd if and only if either
(1) $h^+_{L_0}$ is odd and $t = 1$, or
(2) $h_{L_0}$ is odd, $h^+_{L_0} = 2h_{L_0}$ and $t = 0$.

Conclusion: taking $L_0$ to be $F_0 = \mathbb{Q}(\zeta_m + \zeta_m^{-1})$.
For the prime powers $3, 5, 7, 9, 11, 13, 17, 19, 25, 27$,
$F_0$ has narrow class number 1 and units of independent signs.

For not prime powers $15, 21, 33, 35, 45$,
$F_0$ has narrow class number 2 and units of almost independent signs.

# Main result on integral Shimura data for $m$ prime

Suppose $F = \mathbb{Q}(\zeta_m)$ has class number 1. Let $\gamma = (m, N, a)$.
For simplicity, let $m$ be an odd prime (e.g. $m = 3, 5, 7, 11, 13, 17, 19$).

Let $N \geq 3$ and let $a$ be an arbitrary inertia type. Let $r = N - 2$.

## LMPT part 1

Then the Hurwitz space $Z_\gamma$ has a distinguished point $P$.

It represents an admissible $\mu_m$-cover $h : \mathcal{C} \to T$,
where $T$ is a tree of $r$ projective lines

and $\mathcal{C}$ is a curve of compact type, with $r$ irreducible components $\mathcal{C}_1, \ldots, \mathcal{C}_r$, each of which is a curve of genus $(m-1)/2$ admitting a $\mu_m$-cover of $\mathbb{P}^1$ branched at 3 points.

# Main result on integral Shimura data for $m$ prime

Let $A_j = \mathrm{Jac}(\mathcal{C}_j)$. Then each $A_j$ is a p.p. abelian variety of dimension $(m-1)/2$ with $\mathrm{End}(A_j) \simeq \mathcal{O}_F$.

Let $\Phi_j$ be the CM-type of $A_j$. **Assume** $(F, \Phi_j)$ is simple for each $j$.

This condition is automatic for $m = 3, 5, 11, 17$ (also for $m = 9, 25, 27$).

Let $\beta_j \in \mathbb{C}$ be the value defined earlier for $\Phi_j$.

## LMPT part 2

The integral Shimura datum for the component of $S$ containing $Z_\gamma$ is:

- the $F$-vector space $V = F^r$, together with the standard $\mathcal{O}_F$-lattice $\Lambda = (\mathcal{O}_F)^r \subseteq V$;
- the Hermitian form $\langle \cdot, \cdot \rangle$ on $V$, which takes integral values on $\Lambda$, defined by

$$\langle x, y \rangle = \mathrm{tr}_{F/\mathbb{Q}}(xB\bar{y}^T) \text{ for } B = \mathrm{diag}[\beta_1, \ldots \beta_r] \in \mathrm{GL}_r(F) = \mathrm{GL}(V).$$

# Explanation of part 1: Clutching when $m$ prime

For $i = 1,2$, let $C_i \to \mathbb{P}^1$ be a $\mu_m$-cover, branched at $N_i$ points.
Clutch $C_1$ and $C_2$ together, by identifying last (resp. first) ramified point of $C_1$ (resp. $C_2$) in an ordinary double point.

Get $\mu_m$-cover $C_3 \to \mathbb{P}^1$, with $C_3$ singular.

BLR: $\mathrm{Jac}(C_3) \simeq \mathrm{Jac}(C_1) \times \mathrm{Jac}(C_2)$, with product polarization.

If **admissible**, meaning that $a_1(N) \equiv -a_2(1) \bmod m$,
then $C_3 \to \mathbb{P}^1$ deforms to a $\mu_m$-cover of smooth curves.

Conversely, for $N \geq 4$, Hurwitz spaces are affine so $Z_\gamma$ contains a distinguished point that arises from clutching and represents cover of curves of compact type.

# Example: $m = 5$, $N = 4$, and $a = (1,2,3,4)$

Let $Z_\gamma$ be the Hurwitz family for $\gamma = (5,4,(1,2,3,4))$. Recall $\beta_1 = -\frac{5}{\zeta_5^3 - \zeta_5^2}$
The family $Z_\gamma$ has a distinguished point as follows:

For $j = 1,2$, let $C_j \to \mathbb{P}^1$ be a $\mu_5$-cover with $N = 3$ s.t.
when $j = 1$, let $a = (1,2,2)$. This has $f = (1,0,1,0)$ and $\Phi = \{\sigma_1, \sigma_3\}$;
when $j = 2$, let $a = (3,3,4)$. This has $f = (0,1,0,1)$ and $\Phi = \{\sigma_2, \sigma_4\}$.

The p.p. on $\mathrm{Jac}(C_j)$ is given by $-\beta_1$ for $j = 1$ and by $\beta_1$ for $j = 2$.

The integral Shimura data for $S_\gamma$ has lattice $\mathcal{O}_F^2$ and Hermitian form given by the matrix $B = \mathrm{diag}[-\beta_1, \beta_1]$.

Remark: there is a second distinguished point of $Z_\gamma$ and its Hermitian form is equivalent to this one.

# The simple condition

Not every type is simple:
e.g., $(\mathbb{Q}(\zeta_7), \{\sigma_1, \sigma_2, \sigma_4\})$ is not simple since it is induced from $\mathbb{Q}(\sqrt{-7})$.

Let $m$ be an odd prime power and $F = \mathbb{Q}(\zeta_m)$.

Since $(\mathbb{Z}/m\mathbb{Z})^*$ is cyclic, complex conjugation is the unique automorphism of $F$ of order 2.

The CM-fields in $F$ are the fixed fields $F^H$ for $H \subset (\mathbb{Z}/m)^*$ of odd order.
A CM-type $\Phi$ of $F$ is induced from $F^H$ iff $\Phi$ is a union of cosets of $H$.

Similar issues occur when $m = 13$ or $m = 19$.

# When is life simple?

Let $F = \mathbb{Q}(\zeta_m)$ and let $\Phi$ be a CM-type for $F$.

Suppose $m = 4$ or $m$ is a Fermat prime or twice a Fermat prime.
(e.g., $m = 4$ and $m = 3, 5, 17$ and $m = 6, 10$).
Then $(F, \Phi)$ is simple because $F$ has no proper non-trivial CM-fields.

Suppose $m = 9$ or $m \equiv 3 \bmod 8$ is prime s.t. $w = (m-1)/2$ is prime.
(e.g. $m = 9$ and $m = 11, 59, 83, 107, 179, 227, \ldots$).
Then $\Phi$ is simple.
Reason: The squares are the unique $H \subset (\mathbb{Z}/m\mathbb{Z})^*$ of odd order.
$\Phi$ is not a union of cosets of $H$ iff it contains both a square and a non-square.
Show $\Phi$ contains 1 and either 2 or $-4$.
Under the conditions, $-1$ and 2 are not squares mod $m$, so $\Phi$ is simple.

A similar computation shows $(F, \Phi)$ is simple when $m = 25$ or $m = 27$.

# Integral Shimura data for all cases when $m = 3$

Let $m = 3$ and $F = \mathbb{Q}(\zeta_3)$. Let $N \geq 4$ and $g \geq 2$.

Let $(f_1, f_2)$ be an arbitrary trielliptic signature for $g$.
(meaning $f_1 + f_2 = g$ and $0 \leq \max(f_1, f_2) \leq 2\min(f_1, f_2) + 1$).
Inertia type $a$ has 1 (resp. 2) with mult. $2f_1 - f_2 + 1$ (resp. $2f_2 - f_1 + 1$).

Let $\beta_1 = -\sqrt{-3}$ and $\beta_2 = \sqrt{-3}$.

**Example: all families with $m = 3$**

Let $B \in \mathrm{GL}_g(\mathcal{O}_F)$ be diagonal with $f_1$ entries of $\beta_2$ and $f_2$ entries of $\beta_1$.

For the family $Z_\gamma = Z(3, N, a)$ with signature $(f_1, f_2)$, let $S$ be the component of the Shimura variety containing the Torelli image of $Z_\gamma$.

Then the integral Shimura datum of $S$ has lattice $(\mathcal{O}_F)^g$ with Hermitian form defined by $\langle x, y \rangle = \mathrm{tr}_{F/Q}(xB\bar{y}^T)$.

This includes the three special families M[3], M[6], and M[10].

# Examples when $m = 5$

Let $F = \mathbb{Q}(\zeta_m)$.
For $m = 5$, let $\beta_1 = -\frac{5}{\zeta_5^3 - \zeta_5^2}$ and $\beta_2 = \sigma_2(\beta_1)$.
For the two Moonen special families $M = Z(m, N, a)$ below, let $r = N - 2$, and with $B \in \mathrm{GL}_r(\mathcal{O}_F)$ as given, the integral Shimura datum of $M$ is the lattice $(\mathcal{O}_F)^r$ together with the Hermitian form

$$\langle x, y \rangle = \mathrm{tr}_{F/Q}(xB\bar{y}^T).$$

| $M$ | $(m,N,a)$ | $B$ |
|---|---|---|
| $M[11]$ | $(5,4,(1,3,3,3))$ | $\mathrm{diag}[\beta_1, \beta_2]$ |
| $M[16]$ | $(5,5,(2,2,2,2,2))$ | $\mathrm{diag}[-\beta_1, -\beta_2, -\beta_2]$ |

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As a short-hand, this is the admissible distinguished point:

| $M$ | $(m,N,a)$ | distinguished |
|---|---|---|
| $M[11]$ | $(5,4,(1,3,3,3))$ | $(1,3,1)+(4,3,3)$ |
| $M[16]$ | $(5,5,(2,2,2,2,2))$ | $(2,2,1)+(4,2,4)+(1,2,2)$ |

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# Generalizations

We want to generalize to other $m$ s.t. $F = \mathbb{Q}(\zeta_m)$ has class number 1.

So far, we can find integral Shimura data under conditions on the inertia type $a$.

**Complications**

(i) Some families do not have any distinguished points of compact type. E.g., $M[12]$, with $m = 6$, $N = 4$, and $a = (1,1,1,3)$.

(ii) Some distinguished points of compact type involve induced covers. E.g. $M[18]$ when $m = 10$, $N = 4$, and $a = (3,5,6,6)$.
(The cover is disconnected over a component of the tree)

(iii) Some distinguished points involve non-simple CM-types
E.g. $M[17]$ when $m = 7$, $N = 4$, and $a = (2,4,4,4)$.

# Preview of future work

Despite these complications, we can handle families with arbitrary number of branch points $N$

[A] when $m = 2m'$ (e.g. $m = 6, 10, 22, 34$)
E.g., we found the integral Shimura data for M[18], where $m = 10$, $N = 4$, and $a = (3, 5, 6, 6)$. Here $g = 6$ and $f = (1, 1, 0, 1, 0, 0, 2, 0, 1)$.

[B] when $m$ is a prime power (e.g. $m = 9, 25, 27$) and

[C] when the family has a point representing a curve with extra automorphisms (e.g., $m = 7, 13, 19, 8, 16$)
E.g., we found the Shimura data for $M[17]$, where $m = 7$, $N = 4$, and $a = (2, 4, 4, 4)$. Here $g = 6$ and $f = (1, 2, 0, 2, 0, 1)$.

Remark: our results for [C] use more algebraic number theory and circumvent the problem (iii) of non-simple CM-types.

# Thanks!