Lifting low-gonal curves for use in Tuitman's algorithm

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- Tuitman: arbitrary* curves $\overline{C}$ equipped with a map $\overline{\varphi} : \overline{C} \to \mathbb{P}^1$.
- Tuitman's algorithm requires a *lift* of $(\overline{C}, \overline{\varphi})$ to (C, φ) defined over K with some technical conditions.

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with $d \leq 5$ and denote by $\overline{C}$ the non-singular model. Let $\overline{\varphi}$ be projection onto x and assume that $\overline{\varphi}$ is simply branched.

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- the reduction of φ mod p is $\overline{\varphi}$.

Reduced bases

Let $\mathbb{F}_q[\overline{C}]_0$ (resp. $\mathbb{F}_q[\overline{C}]_\infty$) be the integral closure of $\mathbb{F}_q[x]$ (resp. $\mathbb{F}_q[1/x]$) inside $\mathbb{F}_q(\overline{C})$.

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Theorem (Hess)

Let k be a field and $k(C)$ a degree d function field. There exist unique negative integers $r_1 \geq r_2 \geq \ldots \geq r_{d-1}$ for which there is a basis $1, \alpha_1, \ldots, \alpha_{d-1}$ of $k[C]_0$ over $k[x]$ such that $1, x^{r_1}\alpha_1, \ldots, x^{r_{d-1}}\alpha_{d-1}$ is a basis of $k[C]_\infty$ over $k[1/x]$.

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Compare with Minkowski reduced bases. We call $e_i = -r_i - 2$ the *Maroni invariants* of $\overline{C}$ with respect to $\overline{\varphi}$. A corresponding basis is called a *reduced basis*. We have

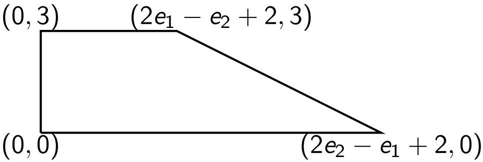
- $-1 \leq e_1 \leq ... \leq e_{d-1} \leq \frac{2g-2}{d}$,
- $e_1 + ... + e_{d-1} = g - d + 1$.

Liftable models in degree $d = 3$

There is a model of $\overline{C}$ of the form

$$\overline{f}_3(x)y^3 + \overline{f}_2(x)y^2z + \overline{f}_1(x)yz^2 + \overline{f}_0(x)z^3 = 0$$

inside $\mathbb{A}^1 \times \mathbb{P}^1$ with Newton polygon ($z = 1$)



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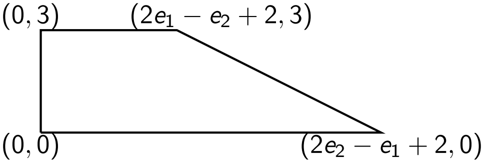

Such a model can be lifted naively to $\mathcal{O}_K$. How to compute this explicitly?

The Delone–Faddeev correspondence

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Theorem (Delone, Faddeev)

There is a canonical bijection between cubic rings R over $\mathbb{F}_q[x]$, up to isomorphism, and binary cubic forms over $\mathbb{F}_q[x]$, up to an action of $\mathrm{GL}_2(\mathbb{F}_q[x])$.

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- The bijection is very explicit and can be done on a computer.

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There is a model of $\overline{C}$ in $\mathbb{A}^1 \times \mathbb{P}^2$ defined as a complete intersection by

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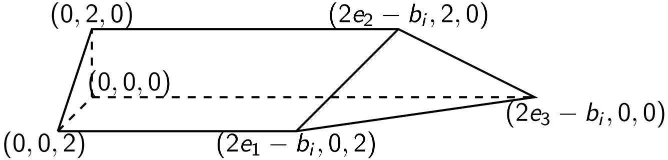

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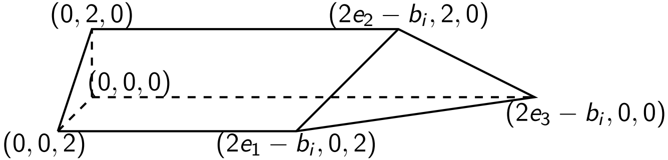

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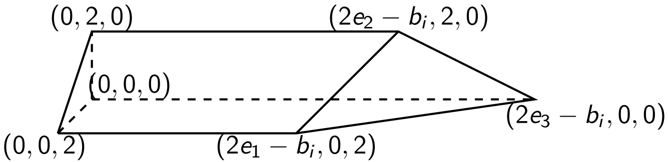

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The Bhargava correspondence

Theorem (Bhargava)

There is a canonical bijection between pairs (R, S) *where* R *is a quartic ring over* $\mathbb{F}_q[x]$ *and* S *is a cubic resolvent of* R*, up to isomorphism, and pairs* (Q_1, Q_2) *of ternary quadratic forms over* $\mathbb{F}_q[x]$*, up to an action of* $\mathrm{GL}_3(\mathbb{F}_q[x]) \times \mathrm{GL}_2(\mathbb{F}_q[x])$.

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- Change of basis of R (resp. S) corresponds to action of GL_3 (resp. GL_2).
- The cubic resolvent of $\mathbb{F}_q[\overline{C}]_0$ is of the form $\mathbb{F}_q[\overline{C}']_0$ for some cubic function field $\mathbb{F}_q(\overline{C}')/\mathbb{F}_q(x)$.

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- Works well in practice, running time is dominated by Tuitman's algorithm.
- $d \geq 7$ is impossible by the non-unirationality of the Hurwitz spaces $\mathcal{H}_{d,g}$. Degree $d = 6$ is not known.
- Computing these liftable models is possible over many fields, not just finite fields.