Modular curves of prime-power level with infinitely many rational points

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Galois representations

Let E be an elliptic curve over $\mathbb{Q}$ and let $N \geqslant 1$ be an integer.

The Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on the N-torsion subgroup of $E(\overline{\mathbb{Q}})$,

$$E[N] \simeq \mathbb{Z}/N\mathbb{Z} \oplus \mathbb{Z}/N\mathbb{Z},$$

via its action on points (coordinate-wise). This yields a representation

$$\rho_{E,N}\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Aut}(E[N]) \simeq \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}),$$

whose image we denote $G_E(N)$. Choosing bases compatibly, we can take the inverse limit and obtain a single representation

$$\rho_E\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \varprojlim_N \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}) \simeq \mathrm{GL}_2(\hat{\mathbb{Z}}),$$

whose image we denote G_E, with projections $G_E \to G_E(N)$ for each N.

Modular curves

Let $F_N \coloneqq \mathbb{Q}(\zeta_n)(X(N))$. Then $F_1 = \mathbb{Q}(j)$ and $F_N/\mathbb{Q}(j)$ is Galois with

$$\mathrm{Gal}(F_N/\mathbb{Q}(j)) \simeq \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I\}$$

Let $G \subseteq \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ be a group containing $-I$ with $\det(G) = (\mathbb{Z}/N\mathbb{Z})^\times$.
Define $X_G/\mathbb{Q}$ to be the smooth projective curve with function field F_N^G.
Let $J_G \colon X_G \to X(1) = \mathbb{Q}(j)$ be the map corresponding to $\mathbb{Q}(j) \subseteq F_N^G$.

If $M|N$ and G is the full inverse image of $H \subseteq \mathrm{GL}_2(\mathbb{Z}/M\mathbb{Z})$, then $X_G = X_H$. We call the least such M the *level* of G and X_G.

Better: identify G with $\pi_N^{-1}(G)$, where $\pi_N \colon \mathrm{GL}_2(\hat{\mathbb{Z}}) \to \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$;
G as an open subgroup of $\mathrm{GL}_2(\hat{\mathbb{Z}})$ containing $-I$ with $\det(G) = \hat{\mathbb{Z}}^\times$.

For any $E/\mathbb{Q}$ with $j(E) \notin \{0, 1728\}$, up to $\mathrm{GL}_2(\hat{\mathbb{Z}})$-conjugacy,

$$G_E \subseteq G \iff j(E) \in J_G(X_G(\mathbb{Q})).$$

Congruence subgroups

For $G \subseteq \mathrm{GL}_2(\hat{\mathbb{Z}})$ of level N as above, let $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ be the preimage of $\pi_N(G) \cap \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$.

Then Γ is a congruence subgroup containing $\Gamma(N)$, and the modular curve $X_\Gamma \coloneqq \Gamma\backslash\mathfrak{h}^*$ is isomorphic to the base change of X_G to $\mathbb{Q}(\zeta_n)$.

The genus g of X_G and X_Γ must coincide, but their levels need not (!); the level M of X_Γ may strictly divide the level N of X_G.

For each $g \geqslant 0$ we have $g(X_\Gamma) = g$ for only finitely many X_Γ;
for $g \leqslant 24$ these Γ can be found in the tables of Cummins and Pauli.

But we may have $g(X_G) = g$ for infinitely many X_G (!)

Call $g(X_G)$ the genus of G.

Restricting the level

For each odd prime p there is a $G \subseteq \mathrm{GL}_2(\mathbb{Z}/2p\mathbb{Z})$ of index 2 that surjects on to both $\mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z})$ and $\mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$.

The corresponding X_G are all genus 0 curves isomorphic to $\mathbb{P}^1_{\mathbb{Q}}$ with maps $J_G\colon X_G \to X(1)$ given by $J_G(t) = 1728 + \left(\frac{-1}{p}\right)pt^2$.

One can similarly construct infinite families of genus 0 curves $X_G \simeq \mathbb{P}^1_{\mathbb{Q}}$ of level N divisible by any of $3, 5, 7, 13$.

And one can take combinations. For example, there are 12 non-conjugate G of level 91 and index at least 24 for which $X_G \simeq \mathbb{P}^1_{\mathbb{Q}}$.

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So we want to restrict N.

Theorem: For prime N, there are 29 X_G with $X_G(\mathbb{Q})$ infinite.
Conjecture: For prime N, there are 64 X_G with $X_G(\mathbb{Q}) \neq \emptyset$.

Now let N be a prime power.

Main results

Theorem

There are 248 modular curves X_G of prime power level with $X_G(\mathbb{Q})$ infinite. Of these, 220 have genus 0 and 28 have genus 1.

For each of these 248 groups G we have an explicit $J_G\colon X_G \to X(1)$.

The 201 of 2-power level were independently found by Jeremy Rouse and David Zureick-Brown [RZB15].[1]

Corollary

For each of these G we can completely describe the set of j-invariants of elliptic curves $E/\mathbb{Q}$ for which $G_E \subseteq G$.

Corollary

There are 1294 non-conjugate open subgroups of $\mathrm{GL}_2(\hat{\mathbb{Z}})$ of prime power level that occur as G_E for infinitely many $E/\mathbb{Q}$ with distinct $j(E)$.

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[1] **Note**: our G act on the left and are thus transposed relative to those in [RZB15].

Finiteness results

Call an open subgroup $G \subseteq \mathrm{GL}_2(\hat{\mathbb{Z}})$ *admissible* if

- $\det(G) = \hat{\mathbb{Z}}^\times$ and $-I \in G$;
- G contains an element $\mathrm{GL}_2(\hat{\mathbb{Z}})$-conjugate to $\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$ or $\left(\begin{smallmatrix}1 & 1\\ 0 & -1\end{smallmatrix}\right)$.

If $G_E \subseteq G$ for some $E/\mathbb{Q}$ then G must be admissible.

Proposition

For any fixed $g \geqslant 0$, only finitely many admissible G have genus $\leqslant g$ and prime power level.

Proposition

There are 220 admissible G of prime power level and genus 0.
There are 250 admissible G of prime power level and genus 1.

Distinguishing finite from infinite when $g = 0$

Let $G \subseteq \mathrm{GL}_2(\hat{\mathbb{Z}})$ be admissible of ℓ-power level and genus $g = 0$.

Since $g = 0$ it is enough to show $X_G(\mathbb{Q}) \neq \emptyset$.

Distinguishing finite from infinite when $g = 0$

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Since $g = 0$ it is enough to show $X_G(\mathbb{Q}) \neq \emptyset$.

But this is necessarily true!

Our criterion for admissibility guarantees $X_G(\mathbb{R}) \neq \emptyset$.

For $p \nmid \ell$ we must have $X_G(\mathbb{Q}_p) \neq \emptyset$, since $\overline{X}_G(\mathbb{F}_p) \neq \emptyset$ (by Hensel).

Thus $X_G(\mathbb{Q}_\ell) \neq \emptyset$ (by Hilbert).

Therefore $X_G(\mathbb{Q}) \neq \emptyset$ (by Hasse).

Distinguishing finite from infinite when $g = 1$

If $g = 1$ then either $X_G(\mathbb{Q}) = \emptyset$, or $X_G(\mathbb{Q})$ can be identified with an elliptic curve E_G whose conductor is a power of ℓ.

There are only finitely many such E_G, all in Cremona's tables. For a finite set S of small primes $p \nmid \ell$, their a_p-values distinguish their isogeny classes, and therefore their ranks.

For each of these $p \nmid \ell$ we compute $a_p = p + 1 - \#X_G(\mathbb{F}_p)$ where

$$\#X_G(\mathbb{F}_p) = \sum_{j \in \mathbb{F}_p} \#\{P \in Y_G(\mathbb{F}_p) : J_G(P) = j\} + \#X_G^\infty(\mathbb{F}_p)$$

We compute the sum using the moduli interpretation of $Y_G/\mathbb{F}_p$. We don't need a model for X_G, we only need G.

If no match, $X_G(\mathbb{Q}) = \emptyset$. If rank 0 match, $X_G(\mathbb{Q})$ is finite. If rank > 0 match, either $X_G(\mathbb{Q}) = \emptyset$ or $X_G(\mathbb{Q})$ is infinite. To distinguish which applies it suffices to find $E/\mathbb{Q}$ with $G_E \subseteq G$.

We find that only 28 of the 250 are elliptic curves of rank $r > 0$.

#	group	i	N	generators	map	sup
0	1A^0-1a	1	1			
1	3A^0-3a	3	3	$[0,1,2,0], [1,1,1,2], [1,0,0,2]$	t^3	0
2	3B^0-3a	4	3	$[0,1,2,1], [1,2,0,2]$	$(t+3)^3(t+27)/t$	0
3	3C^0-3a	6	3	$[0,1,2,0], [1,0,0,2]$	$(t-9)(t+3)/t$	1
4	3D^0-3a	12	3	$[2,0,0,2], [1,0,0,2]$	$729/(t^3-27)$	2
					$-27(t-3)/(t^2+3t+9)$	3
5	9A^0-9a	9	9	$[0,2,4,0], [1,1,4,5], [1,0,0,2]$	t^3+9t-6	1
6	9B^0-9a	12	9	$[1,1,0,1], [2,0,0,5], [1,0,0,2]$	$t(t^2+9t+27)$	2
7	9C^0-9a	12	9	$[2,0,0,5], [4,2,3,4], [1,0,0,2]$	t^3	2
8	9D^0-9a	18	9	$[2,0,0,5], [1,3,3,1], [0,2,4,0], [1,0,0,2]$	$-27/t^3$	3
					$(t^2-3)/t$	5
9	9E^0-9a	18	9	$[1,3,0,1], [2,1,1,1], [4,2,0,5]$	$-9(t^3+3t^2-9t-3)/(8t^3)$	1
10	9F^0-9a	27	9	$[0,2,4,1], [4,3,5,4], [4,5,0,5]$	$\dfrac{3^7(t^2-1)^3(t^6+3t^5+6t^4+t^3-3t^2+12t+16)^3}{(2t^3+3t^2-3t-5)^{-1}(t^3-3t-1)^9}$	0
11	9G^0-9a	27	9	$[0,4,2,3], [5,1,1,4], [5,3,0,4]$	$\dfrac{(t^3-9t-12)(9-3t^3)(5t^3+18t^2+18t+3)}{(t^3+3t^2-3)^3}$	1
12	9H^0-9a	36	9	$[1,3,0,1], [5,0,3,2], [1,0,2,2]$	$3(t^3+9)/t^3$	4
					$3t/(2t^2-3t+6)$	9
13	9H^0-9b	36	9	$[1,3,0,1], [5,0,3,2], [2,1,0,1]$	$3(t^3+9t^2-9t-9)/(t^3-9t^2-9t+9)$	4
14	9H^0-9c	36	9	$[1,3,0,1], [5,0,3,2], [4,2,0,5]$	$-6(t^3-9t)/(t^3+9t^2-9t-9)$	4
					$-(t^2+3)/(t^2+8t+3)$	10
15	9I^0-9a	36	9	$[2,1,0,5], [1,2,3,2]$	$-6(t^3-9t)/(t^3-3t^2-9t+3)$	6
16	9I^0-9b	36	9	$[2,1,0,5], [4,0,3,5]$	$-3(t^3+9t^2-9t-9)/(t^3+3t^2-9t-3)$	6
17	9I^0-9c	36	9	$[2,2,0,5], [2,2,3,1]$	$(t^3-6t^2+3t+1)/(t^2-t)$	6
18	9J^0-9a	36	9	$[1,3,0,1], [2,2,3,8], [1,2,0,2]$	$(t^3-3t+1)/(t^2-t)$	7
19	9J^0-9b	36	9	$[1,3,0,1], [2,2,3,8], [2,1,0,1]$	$-18(t^2-1)/(t^3-3t^2-9t+3)$	7
20	9J^0-9c	36	9	$[1,3,0,1], [5,2,3,5], [4,0,0,5]$	$3(t^3+3t^2-9t-3)/(t^3-3t^2-9t+3)$	7
21	27A^0-27a	36	27	$[1,1,0,1], [2,1,9,5], [1,2,3,2]$	t^3	6

group	i	N	generators	map	supergroup
1A^0-1a	1	1			
5A^0-5a	5	5	$[2,1,0,3]$, $[1,2,2,0]$, $[1,1,0,2]$	$t^3(t^2+5t+40)$	1A^0-1a
5B^0-5a	6	5	$[2,0,0,3]$, $[1,0,1,1]$, $[1,0,0,2]$	$(t^2+10t+5)^3/t$	1A^0-1a
5C^0-5a	10	5	$[3,1,0,2]$, $[1,2,2,0]$, $[2,2,2,1]$	$8000t^3(t+1)(t^2-5t+10)^3/(t^2-5)^5$	1A^0-1a
5D^0-5a	12	5	$[4,0,1,4]$, $[1,0,0,2]$	$125t/(t^2-11t-1)$	5B^0-5a
5D^0-5b	12	5	$[4,0,1,4]$, $[2,0,0,1]$	$(t^2-11t-1)/t$	5B^0-5a
5E^0-5a	15	5	$[2,1,0,3]$, $[2,0,2,3]$, $[1,0,2,2]$	$(t+5)(t^2-5)/(t^2+5t+5)$	5A^0-5a
5G^0-5a	30	5	$[3,1,0,2]$, $[2,1,0,1]$	$125/(t(t^4+5t^3+15t^2+25t+25))$	5E^0-5a
				$(t^2+5)/t$	5B^0-5a
5G^0-5b	30	5	$[3,1,0,2]$, $[2,1,3,3]$	$-t(t^2+5t+10)/(t^3+5t^2+10t+10)$	5C^0-5a
				$-5(t^2+4t+5)/(t^2+5t+5)$	5E^0-5a
5H^0-5a	60	5	$[4,0,0,4]$, $[2,0,0,1]$	$-1/t^5$	5D^0-5a
				$\frac{-(t^4-2t^3+4t^2-3t+1)}{t(t^4+3t^3+4t^2+2t+1)}$	5D^0-5b
				$5t/(t^2-t-1)$	5G^0-5a
25A^0-25a	30	25	$[2,2,0,13]$, $[4,1,3,1]$, $[2,3,0,6]$	$(t-1)(t^4+t^3+6t^2+6t+11)$	5B^0-5a
25B^0-25a	60	25	$[9,10,0,14]$, $[0,7,7,2]$, $[2,8,0,1]$	$-t^5$	5D^0-5b
				$(1-t^2)/t$	25A^0-25a
25B^0-25b	60	25	$[9,10,0,14]$, $[0,7,7,2]$, $[4,1,0,7]$	$\frac{-(t^4-2t^3+4t^2-3t+1)}{t(t^4+3t^3+4t^2+2t+1)}$	5D^0-5a
				$(t^2+4t-1)/(t^2-t-1)$	25A^0-25a
7B^0-7a	8	7	$[2,0,0,4]$, $[3,0,1,5]$, $[1,0,0,3]$	$(t^2+5t+1)^3(t^2+13t+49)/t$	1A^0-1a
7D^0-7a	21	7	$[0,3,2,3]$, $[2,4,4,5]$, $[3,1,0,4]$	$\frac{(2t-1)^3(t^2-t+2)^3(2t^2+5t+4)^3(5t^2+2t-4)^3}{(t^3+2t^2-t-1)^7}$	1A^0-1a
7E^0-7a	24	7	$[6,0,1,6]$, $[1,0,0,3]$	$49(t^2-t)/(t^3-8t^2+5t+1)$	7B^0-7a
7E^0-7b	24	7	$[6,0,1,6]$, $[3,0,0,1]$	$(t^3-8t^2+5t+1)/(t^2-t)$	7B^0-7a
7E^0-7c	24	7	$[6,0,1,6]$, $[3,0,0,4]$	$-7(t^3-2t^2-t+1)/(t^3-t^2-2t+1)$	7B^0-7a
7F^0-7a	28	7	$[3,1,4,4]$, $[4,4,1,3]$, $[3,4,0,4]$	$\frac{t(t+1)^3(t^2-5t+1)^3(t^2-5t+8)^3}{(t^4-5t^3+8t^2-7t+7)^{-3}(t^3-4t^2+3t+1)^7}$	1A^0-1a
13A^0-13a	14	13	$[2,0,0,7]$, $[1,0,1,1]$, $[1,0,0,2]$	$(t^2+5t+13)(t^4+7t^3+20t^2+19t+1)^3/t$	1A^0-1a
13B^0-13a	28	13	$[3,0,0,9]$, $[4,0,1,10]$, $[1,0,0,2]$	$13t/(t^2-3t-1)$	13A^0-13a
13B^0-13b	28	13	$[3,0,0,9]$, $[4,0,1,10]$, $[2,0,0,1]$	$(t^2-3t-1)/t$	13A^0-13a
13C^0-13a	42	13	$[5,0,0,8]$, $[1,0,1,1]$, $[1,0,0,2]$	$13(t^2-t)/(t^3-4t^2+t+1)$	13A^0-13a
13C^0-13b	42	13	$[5,0,0,8]$, $[1,0,1,1]$, $[2,0,0,1]$	$(t^3-4t^2+t+1)/(t^2-t)$	13A^0-13a
13C^0-13c	42	13	$[5,0,0,8]$, $[1,0,1,1]$, $[2,0,0,3]$	$-(5t^3-7t^2-8t+5)/(t^3-4t^2+t+1)$	13A^0-13a

group	i	N	generators	map	supergroup
1A^0-1a	1	1			
2A^0-2a	2	2	$[0,1,1,1]$	t^2+1728	1A^0-1a
2A^0-4a	2	4	$[1,2,0,1]$, $[1,1,1,2]$, $[1,1,0,3]$	$-t^2+1728$	1A^0-1a
2A^0-8a	2	8	$[1,2,0,1]$, $[1,1,1,2]$, $[1,0,0,3]$, $[1,1,0,5]$	$-2t^2+1728$	1A^0-1a
2A^0-8b	2	8	$[1,2,0,1]$, $[1,1,1,2]$, $[1,1,0,3]$, $[1,1,0,5]$	$2t^2+1728$	1A^0-1a
2B^0-2a	3	2	$[0,1,1,0]$	$(256-t)^3/t^2$	1A^0-1a
2C^0-2a	6	2		$-t^2+64$	2B^0-2a
2C^0-4a	6	4	$[1,2,0,1]$, $[3,0,0,3]$, $[1,0,2,1]$, $[1,1,0,3]$	$64(t^2+1)$	2B^0-2a
2C^0-8a	6	8	$[1,2,0,1]$, $[3,0,0,3]$, $[1,0,2,1]$, $[1,0,0,3]$, $[0,1,1,0]$	$32(t^2+2)$	2B^0-2a
2C^0-8b	6	8	$[1,2,0,1]$, $[3,0,0,3]$, $[1,0,2,1]$, $[1,1,0,3]$, $[1,1,0,5]$	$-32(t^2-2)$	2B^0-2a
4A^0-4a	4	4	$[1,1,1,2]$, $[0,1,3,0]$, $[1,1,0,3]$	$4t^3(8-t)$	1A^0-1a
4B^0-4a	6	4	$[3,0,0,3]$, $[0,1,3,2]$, $[1,0,0,3]$	$256/(t^2+4)$	2B^0-2a
4B^0-4b	6	4	$[3,0,0,3]$, $[0,1,3,2]$, $[1,2,0,3]$	$-4096/(t^2+16t)$	2B^0-2a
4B^0-8a	6	8	$[3,0,0,3]$, $[1,4,0,1]$, $[0,3,5,2]$, $[1,0,0,3]$, $[1,2,0,5]$	$-512/(t^2-8)$	2B^0-2a
4B^0-8b	6	8	$[3,0,0,3]$, $[1,4,0,1]$, $[0,3,5,2]$, $[1,2,0,3]$, $[1,2,0,5]$	$512/(t^2+8)$	2B^0-2a
4C^0-4a	6	4	$[1,2,2,1]$, $[0,1,3,0]$, $[1,0,0,3]$	t^2	2B^0-2a
4C^0-4b	6	4	$[1,2,2,1]$, $[0,1,3,0]$, $[1,2,0,3]$	$-t^2$	2B^0-2a
4C^0-8a	6	8	$[1,4,0,1]$, $[2,1,3,2]$, $[0,3,5,0]$, $[1,0,0,3]$, $[1,2,0,5]$	$-128t^2$	2B^0-2a
4C^0-8b	6	8	$[1,4,0,1]$, $[2,1,3,2]$, $[0,3,5,0]$, $[1,2,0,3]$, $[1,2,0,5]$	$128t^2$	2B^0-2a
4D^0-4a	8	4	$[2,1,1,3]$, $[1,1,0,3]$	$-(t^2+2t-2)/t$	4A^0-4a
4D^0-8a	8	8	$[1,4,0,1]$, $[2,1,5,3]$, $[3,3,0,5]$, $[0,1,3,0]$	$(t^2+4t-2)/(4-2t^2)$	4A^0-4a
4E^0-4a	12	4	$[3,0,0,3]$, $[1,2,2,1]$, $[0,1,1,0]$	$(t^2-1)/(2t)$	2C^0-4a
4E^0-4b	12	4	$[3,0,0,3]$, $[1,2,2,1]$, $[1,0,0,3]$	$8(2t^2+1)$	2C^0-2a
4E^0-4c	12	4	$[3,0,0,3]$, $[1,2,2,1]$, $[1,2,0,3]$	$4(t^2+1)/t$	2C^0-2a
4E^0-8a	12	8	$[3,0,0,3]$, $[1,4,0,1]$, $[5,0,0,5]$, $[1,2,2,5]$, $[1,0,0,3]$, $[0,1,1,0]$	$4t/(t^2-2)$	2C^0-8a
4E^0-8b	12	8	$[3,0,0,3]$, $[1,4,0,1]$, $[5,0,0,5]$, $[1,2,2,5]$, $[1,0,0,3]$, $[1,2,0,5]$	$8(t^2-1)$	2C^0-2a
4E^0-8c	12	8	$[3,0,0,3]$, $[1,4,0,1]$, $[5,0,0,5]$, $[1,2,2,5]$, $[1,2,0,3]$, $[0,1,1,0]$	$(t^2-2)/(2t)$	2C^0-8a
4E^0-8d	12	8	$[3,0,0,3]$, $[1,4,0,1]$, $[5,0,0,5]$, $[1,2,2,5]$, $[1,2,0,3]$, $[1,2,0,5]$	$8(t^2+1)$	2C^0-2a
4E^0-8e	12	8	$[3,0,0,3]$, $[1,4,0,1]$, $[5,0,0,5]$, $[1,2,2,5]$, $[1,2,0,5]$, $[0,1,1,0]$	$(t^2+1)/(t^2-2t-1)$	4C^0-8b
4E^0-8f	12	8	$[3,0,0,3]$, $[1,4,0,1]$, $[5,0,0,5]$, $[1,2,2,5]$, $[3,0,0,5]$, $[0,1,3,2]$	$(t^2+2t-1)/(t^2+1)$	2C^0-8b
4E^0-8g	12	8	$[3,0,0,3]$, $[1,4,0,1]$, $[5,0,0,5]$, $[1,2,2,5]$, $[3,0,0,5]$, $[2,1,1,2]$	$4t/(t^2+2)$	2C^0-8b
4E^0-8h	12	8	$[3,0,0,3]$, $[1,4,0,1]$, $[5,0,0,5]$, $[1,2,2,5]$, $[3,2,0,5]$, $[0,1,3,0]$	$2(t^2+1)/(t^2+2t-1)$	2C^0-8b
4E^0-8i	12	8	$[3,0,0,3]$, $[1,4,0,1]$, $[5,0,0,5]$, $[1,2,2,5]$, $[3,2,0,5]$, $[2,1,1,2]$	$(t^2+2)/(2t)$	2C^0-8b

group	i	N	generators	map	super
4F^0-4a	12	4	[0, 1, 3, 0], [1, 0, 0, 3]	$8(t^2-1)$	4C^0-4a
4F^0-4b	12	4	[0, 1, 3, 0], [2, 1, 1, 2]	$8(t^2+1)$	4C^0-4a
4F^0-8a	12	8	[3, 0, 0, 3], [1, 4, 0, 1], [0, 3, 5, 0], [1, 0, 0, 3], [1, 2, 2, 1]	$4(t^2+2)$	4C^0-4a
4F^0-8b	12	8	[3, 0, 0, 3], [1, 4, 0, 1], [0, 3, 5, 0], [3, 0, 0, 5], [1, 2, 2, 1]	$4(t^2-2)$	4C^0-4a
4G^0-16a	24	16	[1, 4, 0, 1], [7, 0, 0, 7], [3, 0, 0, 11], [1, 0, 4, 1], [1, 1, 0, 5], [1, 5, 2, 5]	$(1-t^2)/(2t)$	4E^0-8f
4G^0-4a	24	4	[3, 0, 0, 3], [1, 0, 0, 3]	$1/4t^2$	4E^0-4c
4G^0-4b	24	4	[3, 0, 0, 3], [1, 3, 0, 3]	$t^2/2$	4E^0-4a
4G^0-8a	24	8	[3, 0, 0, 3], [1, 4, 0, 1], [5, 0, 0, 5], [1, 0, 4, 1], [0, 1, 3, 0], [2, 1, 5, 2]	$4t/(t^2-2)$	4F^0-4b
4G^0-8b	24	8	[3, 0, 0, 3], [1, 4, 0, 1], [5, 0, 0, 5], [1, 0, 4, 1], [1, 0, 0, 3], [0, 1, 1, 0]	$2(t^2+2t+2)/(t^2-2)$	4F^0-4a
4G^0-8c	24	8	[3, 0, 0, 3], [1, 4, 0, 1], [5, 0, 0, 5], [1, 0, 4, 1], [1, 0, 0, 3], [1, 2, 0, 5]	$t^2/2$	4E^0-4c
4G^0-8d	24	8	[3, 0, 0, 3], [1, 4, 0, 1], [5, 0, 0, 5], [1, 0, 4, 1], [1, 2, 0, 5], [3, 0, 2, 5]	$2(t^2-1)/(t^2+1)$	4F^0-8b
4G^0-8e	24	8	[3, 0, 0, 3], [1, 4, 0, 1], [5, 0, 0, 5], [1, 0, 4, 1], [1, 3, 0, 3], [1, 2, 0, 5]	t^2	4E^0-4a
4G^0-8f	24	8	[3, 0, 0, 3], [1, 4, 0, 1], [5, 0, 0, 5], [1, 0, 4, 1], [1, 3, 0, 3], [1, 3, 2, 3]	$4t/(t^2+2)$	4F^0-4a
8B^0-8a	12	8	[3, 0, 0, 3], [0, 3, 5, 0], [1, 2, 2, 5], [1, 0, 0, 3], [1, 0, 0, 5]	$16t^2$	4C^0-4a
8B^0-8b	12	8	[3, 0, 0, 3], [0, 3, 5, 0], [1, 2, 2, 5], [1, 0, 0, 3], [1, 4, 0, 5]	$32t^2$	4C^0-4a
8B^0-8c	12	8	[3, 0, 0, 3], [0, 3, 5, 0], [1, 2, 2, 5], [3, 2, 0, 1], [1, 0, 0, 5]	$32t^2$	4C^0-4b
8B^0-8d	12	8	[3, 0, 0, 3], [0, 3, 5, 0], [1, 2, 2, 5], [3, 2, 0, 1], [1, 4, 0, 5]	$16t^2$	4C^0-4b
8C^0-8a	12	8	[3, 0, 0, 3], [5, 0, 0, 5], [0, 3, 5, 2], [1, 2, 0, 3], [1, 0, 0, 5]	$-8t^2$	4B^0-4b
8C^0-8b	12	8	[3, 0, 0, 3], [5, 0, 0, 5], [0, 3, 5, 2], [1, 2, 0, 3], [1, 4, 0, 5]	$-4(t^2+4)$	4B^0-4b
8C^0-8c	12	8	[3, 0, 0, 3], [5, 0, 0, 5], [0, 3, 5, 2], [3, 2, 0, 1], [1, 0, 0, 5]	$-8(t^2+2)$	4B^0-4b
8C^0-8d	12	8	[3, 0, 0, 3], [5, 0, 0, 5], [0, 3, 5, 2], [3, 2, 0, 1], [1, 4, 0, 5]	$-t^2$	4B^0-4b
8D^0-8a	12	8	[2, 1, 3, 2], [0, 3, 5, 0], [1, 0, 0, 3], [1, 0, 0, 5]	$16/(t^2-2)$	4C^0-4a
8D^0-8b	12	8	[2, 1, 3, 2], [0, 3, 5, 0], [1, 0, 0, 3], [1, 4, 0, 5]	$32/(t^2+4)$	4C^0-4a
8D^0-8c	12	8	[2, 1, 3, 2], [0, 3, 5, 0], [1, 4, 0, 3], [1, 0, 0, 5]	$16/(t^2+2)$	4C^0-4a
8D^0-8d	12	8	[2, 1, 3, 2], [0, 3, 5, 0], [1, 4, 0, 3], [1, 4, 0, 5]	$32/(t^2-4)$	4C^0-4a
8E^0-16a	16	16	[3, 4, 0, 11], [2, 3, 3, 5], [3, 3, 0, 5], [0, 1, 3, 0]	$-4t/(t^2+2)$	4D^0-8a
8E^0-16b	16	16	[3, 4, 0, 11], [2, 3, 3, 5], [3, 3, 0, 5], [0, 3, 1, 0]	$-2(t^2-2t+2)/(t^2-4t+2)$	4D^0-8a
8F^0-8a	16	8	[1, 1, 1, 2], [0, 3, 5, 0], [3, 3, 0, 5], [2, 1, 1, 3]	$8(t^4-4t^2-8t-4)/(t^2-2)^2$	4A^0-4a
8G^0-16a	24	16	[1, 2, 0, 1], [7, 0, 0, 7], [3, 0, 0, 11], [1, 0, 8, 1], [3, 1, 0, 5], [1, 2, 2, 1]	$(1-t^2)/(2t)$	4E^0-8e

group	i	N	generators	map	supergroup
8G^0-8a	24	8	[1, 2, 0, 1], [3, 0, 0, 3], [5, 0, 0, 5], [1, 0, 0, 3], [1, 0, 0, 5]	$(t^2-1)/t$	8C^0-8b
8G^0-8b	24	8	[1, 2, 0, 1], [3, 0, 0, 3], [5, 0, 0, 5], [1, 0, 0, 3], [1, 0, 4, 5]	$(t^2+2)/(2t)$	8C^0-8a
8G^0-8c	24	8	[1, 2, 0, 1], [3, 0, 0, 3], [5, 0, 0, 5], [1, 0, 0, 3], [1, 1, 0, 5]	$(t^2-2)/(2t)$	4E^0-8c
8G^0-8d	24	8	[1, 2, 0, 1], [3, 0, 0, 3], [5, 0, 0, 5], [1, 0, 0, 5], [3, 0, 2, 1]	$t^2/2$	4E^0-4b
8G^0-8e	24	8	[1, 2, 0, 1], [3, 0, 0, 3], [5, 0, 0, 5], [1, 0, 2, 3], [3, 2, 2, 1]	t^2	4E^0-4b
8G^0-8f	24	8	[1, 2, 0, 1], [3, 0, 0, 3], [5, 0, 0, 5], [1, 1, 0, 3], [1, 0, 0, 5]	$(t^2-1)/(2t)$	4E^0-4a
8G^0-8g	24	8	[1, 2, 0, 1], [3, 0, 0, 3], [5, 0, 0, 5], [1, 1, 0, 3], [1, 1, 4, 1]	$(t^2+2)/(2t)$	4E^0-8i
8G^0-8h	24	8	[1, 2, 0, 1], [3, 0, 0, 3], [5, 0, 0, 5], [1, 1, 0, 3], [3, 2, 2, 1]	$2(t^2+1)/(t^2+2t-1)$	4E^0-8g
8G^0-8i	24	8	[1, 2, 0, 1], [3, 0, 0, 3], [5, 0, 0, 5], [1, 3, 2, 3], [3, 2, 2, 1]	$(t^2+2)/(2t)$	4E^0-8g
8G^0-8j	24	8	[1, 2, 0, 1], [3, 0, 0, 3], [5, 0, 0, 5], [3, 0, 0, 5], [1, 1, 4, 1]	$2(t^2+1)/(t^2-2t-1)$	4E^0-8i
8G^0-8k	24	8	[1, 2, 0, 1], [3, 0, 0, 3], [5, 0, 0, 5], [3, 1, 0, 5], [1, 0, 4, 5]	$2(t^2-1)/(t^2+1)$	8D^0-8a
8G^0-8l	24	8	[1, 2, 0, 1], [3, 0, 0, 3], [5, 0, 0, 5], [3, 1, 0, 5], [3, 0, 2, 1]	$(t^2-2)/(2t)$	4E^0-8a
8H^0-8a	24	8	[3, 0, 0, 3], [1, 4, 4, 1], [0, 3, 5, 0], [1, 0, 0, 3], [1, 0, 0, 5]	$2t/(t^2-2)$	8B^0-8a
8H^0-8b	24	8	[3, 0, 0, 3], [1, 4, 4, 1], [0, 3, 5, 0], [1, 0, 0, 3], [1, 2, 2, 3]	$4t/(t^2-2)$	4F^0-8a
8H^0-8c	24	8	[3, 0, 0, 3], [1, 4, 4, 1], [0, 3, 5, 0], [1, 0, 0, 3], [1, 4, 0, 5]	$t/(t^2+1)$	8B^0-8b
8H^0-8d	24	8	[3, 0, 0, 3], [1, 4, 4, 1], [0, 3, 5, 0], [1, 0, 0, 5], [1, 2, 2, 3]	$(t^2+1)/(t^2+2t-1)$	8B^0-8a
8H^0-8e	24	8	[3, 0, 0, 3], [1, 4, 4, 1], [0, 3, 5, 0], [1, 4, 0, 3], [1, 0, 0, 5]	$2t/(t^2+2)$	8B^0-8a
8H^0-8f	24	8	[3, 0, 0, 3], [1, 4, 4, 1], [0, 3, 5, 0], [1, 4, 0, 3], [1, 2, 2, 1]	$(t^2-2)/(2t)$	4F^0-8a
8H^0-8g	24	8	[3, 0, 0, 3], [1, 4, 4, 1], [0, 3, 5, 0], [1, 4, 0, 3], [1, 4, 0, 5]	$t/(t^2-1)$	8B^0-8b
8H^0-8h	24	8	[3, 0, 0, 3], [1, 4, 4, 1], [0, 3, 5, 0], [1, 4, 0, 5], [2, 1, 1, 2]	$(t^2-1)/(2t)$	4F^0-4b
8H^0-8i	24	8	[3, 0, 0, 3], [1, 4, 4, 1], [0, 3, 5, 0], [3, 0, 0, 5], [1, 2, 2, 1]	$(t^2+2)/(2t)$	4F^0-8b
8H^0-8j	24	8	[3, 0, 0, 3], [1, 4, 4, 1], [0, 3, 5, 0], [3, 0, 0, 5], [2, 3, 5, 2]	$2(t^2+1)/(t^2+2t-1)$	4F^0-8b
8H^0-8k	24	8	[3, 0, 0, 3], [1, 4, 4, 1], [0, 3, 5, 0], [3, 4, 0, 5], [1, 2, 2, 1]	$(t^2+2t-1)/(t^2+1)$	4F^0-8b
8H^0-8l	24	8	[3, 0, 0, 3], [1, 4, 4, 1], [0, 3, 5, 0], [3, 4, 0, 5], [2, 1, 1, 2]	$4t/(t^2+2)$	4F^0-8b
8I^0-8a	24	8	[7, 0, 0, 7], [0, 3, 5, 2], [1, 4, 0, 5], [1, 6, 0, 3]	$4t^2/(t^2-2)$	8C^0-8d
8I^0-8b	24	8	[7, 0, 0, 7], [0, 3, 5, 2], [3, 2, 0, 1], [1, 4, 0, 5]	$4t^2/(t^2+2)$	8C^0-8d
8I^0-8c	24	8	[7, 0, 0, 7], [0, 3, 5, 2], [3, 2, 0, 1], [5, 4, 0, 1]	$4/(t^2-1)$	8C^0-8d
8I^0-8d	24	8	[7, 0, 0, 7], [0, 3, 5, 2], [5, 2, 0, 3], [5, 4, 0, 1]	$4/(t^2+1)$	8C^0-8d
8J^0-8a	24	8	[3, 2, 0, 3], [5, 2, 0, 5], [1, 2, 4, 1], [1, 0, 0, 3], [1, 0, 0, 5]	$(t^2+2)/t^2$	4E^0-4c
8J^0-8b	24	8	[3, 2, 0, 3], [5, 2, 0, 5], [1, 2, 4, 1], [1, 0, 0, 3], [1, 2, 0, 5]	t^2-1	4E^0-4c
8J^0-8c	24	8	[3, 2, 0, 3], [5, 2, 0, 5], [1, 2, 4, 1], [1, 2, 0, 3], [1, 0, 0, 5]	$t^2/(t^2-2)$	4E^0-4c
8J^0-8d	24	8	[3, 2, 0, 3], [5, 2, 0, 5], [1, 2, 4, 1], [1, 2, 0, 3], [1, 2, 0, 5]	t^2+1	4E^0-4c

group	i	N	generators	map	super
8K^0-16a	24	16	$[1,4,0,1]$, $[7,0,0,7]$, $[0,3,5,0]$, $[3,0,0,5]$, $[1,2,2,1]$	$(t^2+4)/2$	4F^0-8b
8K^0-16b	24	16	$[1,4,0,1]$, $[7,0,0,7]$, $[0,3,5,0]$, $[7,0,0,9]$, $[1,2,2,1]$	$(t^2-4)/2$	4F^0-8b
8K^0-16c	24	16	$[1,4,0,1]$, $[7,0,0,7]$, $[2,3,1,2]$, $[3,0,0,5]$, $[1,2,2,1]$	t^2+2	4F^0-8b
8K^0-16d	24	16	$[1,4,0,1]$, $[7,0,0,7]$, $[2,3,1,2]$, $[7,0,0,9]$, $[1,2,2,1]$	t^2-2	4F^0-8b
8L^0-8a	24	8	$[0,3,5,0]$, $[5,2,2,1]$, $[1,2,0,3]$, $[1,4,0,5]$	$(t^2-2t-1)/(4t)$	8B^0-8d
8L^0-8b	24	8	$[0,3,5,0]$, $[5,2,2,1]$, $[5,4,0,1]$, $[3,6,0,1]$	$2t/(t^2-2t-1)$	8B^0-8d
8N^0-16a	48	16	$[1,4,0,1]$, $[7,0,0,7]$, $[9,0,0,9]$, $[1,0,4,1]$, $[0,1,1,0]$, $[2,3,5,6]$	$(t^2+2)/(2t)$	4G^0-8a
8N^0-16b	48	16	$[1,4,0,1]$, $[7,0,0,7]$, $[9,0,0,9]$, $[1,0,4,1]$, $[0,1,5,0]$, $[2,5,1,2]$	$t^2/2$	4G^0-8e
8N^0-16c	48	16	$[1,4,0,1]$, $[7,0,0,7]$, $[9,0,0,9]$, $[1,0,4,1]$, $[1,2,2,1]$, $[0,5,1,0]$	$4t/(t^2-1)$	8K^0-16c
8N^0-16d	48	16	$[1,4,0,1]$, $[7,0,0,7]$, $[9,0,0,9]$, $[1,0,4,1]$, $[3,2,2,5]$, $[2,3,5,6]$	$2(t^2+1)/(t^2-2t-1)$	4G^0-8a
8N^0-16e	48	16	$[1,4,0,1]$, $[7,0,0,7]$, $[9,0,0,9]$, $[1,0,4,1]$, $[7,0,0,9]$, $[1,2,2,1]$	$(t^2-1)/(2t)$	4G^0-8d
8N^0-16f	48	16	$[1,4,0,1]$, $[7,0,0,7]$, $[9,0,0,9]$, $[1,0,4,1]$, $[7,0,0,9]$, $[2,1,1,6]$	$(t^2-2)/(2t)$	4G^0-8f
8N^0-32a	48	32	$[1,4,0,1]$, $[15,0,0,15]$, $[7,0,0,23]$, $[1,0,4,1]$, $[3,0,0,5]$, $[2,3,1,0]$	$(1-2t-t^2)/(t^2-2t-1)$	4G^0-16a
8N^0-8a	48	8	$[1,4,0,1]$, $[7,0,0,7]$, $[1,0,4,1]$, $[1,2,0,3]$, $[1,0,0,5]$	t^2	4G^0-4a
8N^0-8b	48	8	$[1,4,0,1]$, $[7,0,0,7]$, $[1,0,4,1]$, $[1,2,0,3]$, $[1,2,0,5]$	$(t^2-2)/(2t)$	4G^0-8c
8N^0-8c	48	8	$[1,4,0,1]$, $[7,0,0,7]$, $[1,0,4,1]$, $[1,2,0,3]$, $[3,2,0,5]$	$(t^2+1)/t$	4G^0-4a
8N^0-8d	48	8	$[1,4,0,1]$, $[7,0,0,7]$, $[1,0,4,1]$, $[3,0,0,1]$, $[1,0,2,5]$	$2(t^2+1)/(t^2+2t-1)$	4G^0-8c
8N^0-8e	48	8	$[1,4,0,1]$, $[7,0,0,7]$, $[1,0,4,1]$, $[3,0,0,1]$, $[1,2,0,5]$	$(t^2+2)/(2t)$	4G^0-8c
8N^0-8f	48	8	$[1,4,0,1]$, $[7,0,0,7]$, $[1,0,4,1]$, $[3,0,0,1]$, $[5,0,0,1]$	$(t^2-1)/t$	4G^0-4a
8O^0-16a	48	16	$[1,4,0,1]$, $[7,0,0,7]$, $[3,2,0,11]$, $[1,0,8,1]$, $[1,3,0,5]$, $[5,1,4,3]$	$(1-t^2)/(2t)$	8G^0-8j
8O^0-8a	48	8	$[3,2,0,3]$, $[5,2,0,5]$, $[1,0,0,3]$, $[1,0,4,5]$	$(t^2-2)/(2t)$	8G^0-8b
8O^0-8b	48	8	$[3,2,0,3]$, $[5,2,0,5]$, $[1,0,0,5]$, $[1,0,4,3]$	$(t^2-1)/(2t)$	8G^0-8a
8O^0-8c	48	8	$[3,2,0,3]$, $[5,2,0,5]$, $[1,2,0,3]$, $[1,0,0,5]$	$2t/(t^2-1)$	8J^0-8a
8O^0-8d	48	8	$[3,2,0,3]$, $[5,2,0,5]$, $[1,2,0,3]$, $[1,2,0,5]$	$(t^2+2)/(2t)$	8J^0-8b
8O^0-8e	48	8	$[3,2,0,3]$, $[5,2,0,5]$, $[1,2,0,5]$, $[1,2,4,3]$	$(t^2+1)/(2t)$	8I^0-8a
8O^0-8f	48	8	$[3,2,0,3]$, $[5,2,0,5]$, $[1,3,0,3]$, $[1,0,0,5]$	$t^2/2$	8G^0-8f
8O^0-8g	48	8	$[3,2,0,3]$, $[5,2,0,5]$, $[1,3,0,3]$, $[1,2,0,5]$	t^2	8G^0-8f
8O^0-8h	48	8	$[3,2,0,3]$, $[5,2,0,5]$, $[1,3,0,3]$, $[1,3,4,1]$	$(t^2-2)/(2t)$	8G^0-8g
8O^0-8i	48	8	$[3,2,0,3]$, $[5,2,0,5]$, $[3,0,0,5]$, $[1,2,4,3]$	$(t^2-2)/(t^2-4t+2)$	8J^0-8a
8O^0-8j	48	8	$[3,2,0,3]$, $[5,2,0,5]$, $[3,2,0,5]$, $[1,0,4,3]$	$(t^2+2)/(2t)$	8G^0-8b
8O^0-8k	48	8	$[3,2,0,3]$, $[5,2,0,5]$, $[3,3,0,5]$, $[1,0,4,3]$	$(t^2+4t+2)/(t^2-2)$	8I^0-8b
8O^0-8l	48	8	$[3,2,0,3]$, $[5,2,0,5]$, $[3,3,0,5]$, $[1,1,4,1]$	$(t^2+2)/(2t)$	8G^0-8c
8P^0-8a	48	8	$[3,4,4,3]$, $[0,3,5,0]$, $[3,4,0,1]$, $[1,4,0,5]$	$(t^2+2)/(t^2-4t+2)$	8H^0-8g
8P^0-8b	48	8	$[3,4,4,3]$, $[0,3,5,0]$, $[3,4,0,1]$, $[5,4,0,1]$	$(t^2+1)/(2(t-1))$	8H^0-8g

group	i	N	generators	map	supergroup
16B^0-16a	24	16	$[3,0,0,11]$, $[0,3,5,0]$, $[1,2,2,5]$, $[1,0,0,5]$, $[1,8,0,3]$	$t^2/2$	8B^0-8a
16B^0-16b	24	16	$[3,0,0,11]$, $[0,3,5,0]$, $[1,2,2,5]$, $[1,4,0,5]$, $[5,2,0,3]$	$t^2/2$	8B^0-8d
16B^0-16c	24	16	$[3,0,0,11]$, $[0,3,5,0]$, $[1,2,2,5]$, $[3,0,0,5]$, $[1,8,0,3]$	t^2	8B^0-8a
16B^0-16d	24	16	$[3,0,0,11]$, $[0,3,5,0]$, $[1,2,2,5]$, $[3,6,0,5]$, $[3,4,0,7]$	t^2	8B^0-8d
16C^0-16a	24	16	$[7,0,0,7]$, $[3,8,0,11]$, $[0,3,5,2]$, $[1,2,0,3]$, $[1,4,0,5]$	$t^2/2$	8C^0-8b
16C^0-16b	24	16	$[7,0,0,7]$, $[3,8,0,11]$, $[0,3,5,2]$, $[1,2,0,3]$, $[5,4,0,1]$	t^2	8C^0-8b
16C^0-16c	24	16	$[7,0,0,7]$, $[3,8,0,11]$, $[0,3,5,2]$, $[1,4,0,5]$, $[5,2,0,3]$	$2t^2$	8C^0-8d
16C^0-16d	24	16	$[7,0,0,7]$, $[3,8,0,11]$, $[0,3,5,2]$, $[5,4,0,1]$, $[1,6,0,3]$	t^2	8C^0-8d
16D^0-16a	24	16	$[7,0,0,7]$, $[3,0,0,11]$, $[0,3,5,2]$, $[1,4,0,5]$, $[5,2,0,3]$	t^2-4	8C^0-8d
16D^0-16b	24	16	$[7,0,0,7]$, $[3,0,0,11]$, $[0,3,5,2]$, $[3,2,0,1]$, $[1,4,0,5]$	t^2+4	8C^0-8d
16D^0-16c	24	16	$[7,0,0,7]$, $[3,0,0,11]$, $[0,3,5,2]$, $[3,2,0,1]$, $[5,2,0,3]$	$2(t^2-2)$	8C^0-8d
16D^0-16d	24	16	$[7,0,0,7]$, $[3,0,0,11]$, $[0,3,5,2]$, $[3,6,0,5]$, $[3,4,0,7]$	$2(t^2+2)$	8C^0-8d
16E^0-16a	24	16	$[0,3,5,8]$, $[2,1,3,10]$, $[1,0,0,3]$, $[1,0,0,5]$	$(t^2+4)/2$	8D^0-8a
16E^0-16b	24	16	$[0,3,5,8]$, $[2,1,3,10]$, $[1,0,0,3]$, $[1,8,0,5]$	t^2-2	8D^0-8a
16E^0-16c	24	16	$[0,3,5,8]$, $[2,1,3,10]$, $[1,0,0,5]$, $[1,8,0,3]$	$(t^2-4)/2$	8D^0-8a
16E^0-16d	24	16	$[0,3,5,8]$, $[2,1,3,10]$, $[3,0,0,5]$, $[1,8,0,3]$	t^2+2	8D^0-8a
16F^0-32a	32	32	$[3,4,0,11]$, $[6,3,7,9]$, $[3,3,0,5]$, $[0,3,1,0]$	$-(t^2+2)/(2t)$	8E^0-16b
16F^0-32b	32	32	$[3,4,0,11]$, $[6,3,7,9]$, $[5,5,0,3]$, $[0,3,1,0]$	$(t^2-4t+2)/(t^2-2t+2)$	8E^0-16b
16G^0-16a	48	16	$[1,2,0,1]$, $[7,0,0,7]$, $[3,0,8,11]$, $[1,3,2,3]$, $[3,2,2,1]$	$2(t^2+1)/(t^2+2t-1)$	8G^0-8i
16G^0-16b	48	16	$[1,2,0,1]$, $[7,0,0,7]$, $[3,0,8,11]$, $[3,0,0,1]$, $[1,0,0,5]$	$t^2/2$	8G^0-8a
16G^0-16c	48	16	$[1,2,0,1]$, $[7,0,0,7]$, $[3,0,8,11]$, $[3,0,0,1]$, $[5,0,0,1]$	t^2	8G^0-8a
16G^0-16d	48	16	$[1,2,0,1]$, $[7,0,0,7]$, $[3,0,8,11]$, $[3,0,0,1]$, $[5,1,0,1]$	$(t^2-2)/(2t)$	8G^0-8c
16G^0-16e	48	16	$[1,2,0,1]$, $[7,0,0,7]$, $[3,0,8,11]$, $[3,1,0,1]$, $[1,1,0,5]$	$(t^2+2)/(2t)$	8G^0-8g
16G^0-16f	48	16	$[1,2,0,1]$, $[7,0,0,7]$, $[3,0,8,11]$, $[3,1,0,1]$, $[1,1,4,1]$	$2(t^2+1)/(t^2+2t-1)$	8G^0-8g
16G^0-16g	48	16	$[1,2,0,1]$, $[7,0,0,7]$, $[3,0,8,11]$, $[3,1,0,1]$, $[5,0,0,1]$	$(t^2-1)/(2t)$	8G^0-8f
16G^0-16h	48	16	$[1,2,0,1]$, $[7,0,0,7]$, $[3,0,8,11]$, $[3,2,2,1]$, $[1,4,2,3]$	t^2	8G^0-8e
16G^0-16i	48	16	$[1,2,0,1]$, $[7,0,0,7]$, $[3,0,8,11]$, $[3,2,2,1]$, $[3,3,4,5]$	$(t^2+2)/(2t)$	8G^0-8i
16G^0-16j	48	16	$[1,2,0,1]$, $[7,0,0,7]$, $[3,0,8,11]$, $[3,4,2,1]$, $[1,2,4,5]$	$t^2/2$	8G^0-8e
16G^0-16k	48	16	$[1,2,0,1]$, $[7,0,0,7]$, $[3,0,8,11]$, $[7,1,0,9]$, $[1,2,4,5]$	$(t^2-1)/(2t)$	8G^0-8k
16G^0-16l	48	16	$[1,2,0,1]$, $[7,0,0,7]$, $[3,0,8,11]$, $[7,1,0,9]$, $[3,0,2,1]$	$(t^2-2)/(2t)$	8G^0-8l
16G^0-32a	48	32	$[1,2,0,1]$, $[15,0,0,15]$, $[7,0,0,23]$, $[3,0,8,11]$, $[3,1,0,5]$, $[5,2,2,5]$	$(1-2t-t^2)/(t^2-2t-1)$	8G^0-16a

group	i	N	generators	map	supergroup
$16H^0$-16a	48	16	$[7,0,0,7]$, $[9,0,0,9]$, $[0,3,5,2]$, $[1,4,0,5]$, $[1,2,0,7]$	$4t/(t^2+2)$	$8I^0$-8a
$16H^0$-16b	48	16	$[7,0,0,7]$, $[9,0,0,9]$, $[0,3,5,2]$, $[1,4,0,5]$, $[1,6,0,3]$	$2(t^2+1)/(t^2-2t-1)$	$8I^0$-8a
$16H^0$-16c	48	16	$[7,0,0,7]$, $[9,0,0,9]$, $[0,3,5,2]$, $[1,4,0,5]$, $[5,2,0,3]$	$4t/(t^2-2)$	$8I^0$-8b
$16H^0$-16d	48	16	$[7,0,0,7]$, $[9,0,0,9]$, $[0,3,5,2]$, $[1,6,0,3]$, $[1,2,0,7]$	$(t^2+2t-1)/(t^2+1)$	$8I^0$-8a
$16H^0$-16e	48	16	$[7,0,0,7]$, $[9,0,0,9]$, $[0,3,5,2]$, $[3,2,0,1]$, $[1,2,0,7]$	$(t^2-2)/(2t)$	$16C^0$-16c
$16H^0$-16f	48	16	$[7,0,0,7]$, $[9,0,0,9]$, $[0,3,5,2]$, $[3,2,0,1]$, $[5,2,0,3]$	$(t^2-2)/(2t)$	$8I^0$-8b
$16H^0$-16g	48	16	$[7,0,0,7]$, $[9,0,0,9]$, $[0,3,5,2]$, $[3,2,0,1]$, $[5,4,0,1]$	$(t^2+1)/t$	$16C^0$-16d
$16H^0$-16h	48	16	$[7,0,0,7]$, $[9,0,0,9]$, $[0,3,5,2]$, $[3,4,0,7]$, $[5,6,0,7]$	$(t^2+2)/(2t)$	$16C^0$-16c
$16H^0$-16i	48	16	$[7,0,0,7]$, $[9,0,0,9]$, $[0,3,5,2]$, $[5,2,0,3]$, $[1,6,0,3]$	$(t^2+2t-1)/(t^2+1)$	$16C^0$-16c
$16H^0$-16j	48	16	$[7,0,0,7]$, $[9,0,0,9]$, $[0,3,5,2]$, $[5,2,0,3]$, $[5,4,0,1]$	$2t/(t^2-1)$	$8I^0$-8d
$16H^0$-16k	48	16	$[7,0,0,7]$, $[9,0,0,9]$, $[0,3,5,2]$, $[5,4,0,1]$, $[1,2,0,7]$	$(t^2-1)/t$	$16C^0$-16d
$16H^0$-16l	48	16	$[7,0,0,7]$, $[9,0,0,9]$, $[0,3,5,2]$, $[7,2,0,5]$, $[7,4,0,3]$	$(t^2+2)/(2t)$	$8I^0$-8a
$32A^0$-32a	48	32	$[3,8,0,11]$, $[5,8,0,13]$, $[0,7,9,2]$, $[1,4,0,5]$, $[5,6,0,3]$	$2/t^2$	$16C^0$-16a
$32A^0$-32b	48	32	$[3,8,0,11]$, $[5,8,0,13]$, $[0,7,9,2]$, $[5,2,0,3]$, $[5,4,0,1]$	t^2	$16C^0$-16d
$32A^0$-32c	48	32	$[3,8,0,11]$, $[5,8,0,13]$, $[0,7,9,2]$, $[7,2,0,1]$, $[9,2,0,3]$	t^2	$16C^0$-16a
$32A^0$-32d	48	32	$[3,8,0,11]$, $[5,8,0,13]$, $[0,7,9,2]$, $[7,2,0,5]$, $[7,4,0,3]$	$2/t^2$	$16C^0$-16d

group	i	N	generators	curve	map	supergroup
16C^1-16c	24	16	[2, 1, 3, 2], [0, 3, 5, 8], [1, 0, 0, 5], [1, 8, 0, 3]	256a2	$(x-1)/2$	8D^0-8a
16C^1-16d	24	16	[2, 1, 3, 2], [0, 3, 5, 8], [3, 0, 0, 5], [1, 8, 0, 3]	256a1	$x+1$	8D^0-8a
16B^1-16a	24	16	[3, 0, 0, 11], [0, 3, 5, 0], [2, 3, 9, 6], [1, 0, 0, 3], [1, 0, 0, 5]	256b2	$x/4$	8B^0-8a
16B^1-16c	24	16	[3, 0, 0, 11], [0, 3, 5, 0], [2, 3, 9, 6], [1, 4, 0, 3], [1, 0, 0, 5]	256b1	$x/2$	8B^0-8a
16I^1-16d	48	16	[3, 0, 0, 11], [0, 3, 5, 8], [1, 4, 12, 1], [1, 0, 0, 5], [1, 8, 0, 3]	256a1	$x-1$	8H^0-8a
16I^1-16f	48	16	[3, 0, 0, 11], [0, 3, 5, 8], [1, 4, 12, 1], [1, 8, 0, 3], [2, 3, 5, 2]	256a2	$\frac{4x-4y+12}{x^2-2x-15}$	8H^0-8b
16I^1-16g	48	16	[3, 0, 0, 11], [0, 3, 5, 8], [1, 4, 12, 1], [1, 8, 0, 5], [1, 2, 10, 3]	256a2	$\frac{2(y-x-1)}{x^2-2x-11}$	8H^0-8d
16I^1-16h	48	16	[3, 0, 0, 11], [0, 3, 5, 8], [1, 4, 12, 1], [1, 8, 0, 7], [1, 2, 10, 7]	256a1	$1/x$	8H^0-8j
16I^1-16j	48	16	[3, 0, 0, 11], [0, 3, 5, 8], [1, 4, 12, 1], [3, 0, 0, 5], [1, 8, 0, 3]	256a2	$(x+3)/2$	8H^0-8a
16I^1-16k	48	16	[3, 0, 0, 11], [0, 3, 5, 8], [1, 4, 12, 1], [3, 0, 0, 5], [2, 3, 5, 2]	256a2	$\frac{-x+1}{x+3}$	8H^0-8j
8H^1-16b	48	16	[7, 0, 0, 7], [1, 8, 0, 1], [1, 4, 4, 1], [0, 3, 5, 0], [3, 0, 0, 5], [1, 2, 2, 9]	256a2	$(x+3)/2$	8H^0-8i
8H^1-16c	48	16	[7, 0, 0, 7], [1, 8, 0, 1], [1, 4, 4, 1], [0, 3, 5, 0], [3, 0, 0, 5], [1, 2, 6, 9]	256a2	$\frac{x-1}{x+3}$	8H^0-8j
8H^1-16e	48	16	[7, 0, 0, 7], [1, 8, 0, 1], [1, 4, 4, 1], [0, 3, 5, 0], [7, 0, 0, 9], [1, 2, 2, 1]	256a1	$x-1$	8H^0-8i
8H^1-16g	48	16	[7, 0, 0, 7], [1, 8, 0, 1], [1, 4, 4, 1], [0, 3, 5, 0], [7, 0, 0, 9], [2, 3, 5, 2]	256a1	$-1/x$	8H^0-8j
16D^1-16d	24	16	[3, 8, 0, 11], [0, 3, 5, 0], [5, 2, 2, 1], [3, 2, 0, 5], [5, 4, 0, 1]	128a1	$(x+1)/2$	8B^0-8d
8H^1-16j	48	16	[7, 0, 0, 7], [1, 8, 0, 1], [1, 4, 4, 1], [0, 3, 5, 0], [7, 4, 0, 9], [1, 2, 2, 9]	256a2	$\frac{2(x+y+1)}{x^2-2x-11}$	8H^0-8k
8D^1-16b	24	16	[7, 0, 0, 7], [3, 4, 0, 11], [0, 3, 5, 0], [3, 0, 0, 5], [1, 2, 2, 9]	256a1	$x+1$	4F^0-8b
8H^1-16k	48	16	[7, 0, 0, 7], [1, 8, 0, 1], [1, 4, 4, 1], [0, 3, 5, 0], [7, 4, 0, 9], [1, 2, 6, 9]	256a2	$\frac{4(x-y+3)}{x^2-2x-15}$	8H^0-8l
8D^1-16c	24	16	[7, 0, 0, 7], [3, 4, 0, 11], [0, 3, 5, 0], [3, 4, 0, 5], [1, 2, 2, 1]	256a2	$(x-1)/2$	4F^0-8b
16J^1-16e	48	16	[7, 0, 0, 7], [0, 3, 5, 0], [5, 2, 2, 1], [1, 6, 0, 7], [7, 4, 0, 3]	128a2	$\frac{x^2+2x-7}{8x-8}$	8B^0-8d
16J^1-16g	48	16	[7, 0, 0, 7], [0, 3, 5, 0], [5, 2, 2, 1], [5, 4, 0, 1], [3, 6, 0, 1]	128a2	$(1-x)/2$	8L^0-8b
16F^1-16a	48	16	[3, 0, 0, 11], [1, 4, 4, 1], [0, 3, 5, 0], [1, 0, 0, 3], [1, 0, 0, 5]	256b1	$-2/x$	8H^0-8a
16F^1-16c	48	16	[3, 0, 0, 11], [1, 4, 4, 1], [0, 3, 5, 0], [1, 0, 0, 5], [1, 2, 2, 3]	256b2	$\frac{x^2+2y-8}{x^2-8x+8}$	16B^0-16c
16F^1-16d	48	16	[3, 0, 0, 11], [1, 4, 4, 1], [0, 3, 5, 0], [1, 4, 0, 3], [1, 0, 0, 5]	256b2	$4/x$	8H^0-8e
16F^1-16h	48	16	[3, 0, 0, 11], [1, 4, 4, 1], [0, 3, 5, 0], [3, 0, 0, 5], [1, 2, 2, 1]	256b1	x	8H^0-8i
16F^1-16j	48	16	[3, 0, 0, 11], [1, 4, 4, 1], [0, 3, 5, 0], [3, 4, 0, 1], [1, 2, 2, 1]	256b2	$x/2$	8H^0-8f
16F^1-16k	48	16	[3, 0, 0, 11], [1, 4, 4, 1], [0, 3, 5, 0], [3, 4, 0, 5], [1, 2, 2, 1]	256b2	$\frac{x^2+2y-8}{x^2+4x+8}$	16B^0-16a
11C^1-11a	55	11	[3, 4, 4, 2], [3, 1, 1, 8], [6, 4, 0, 5]	121b1	$J_{11}(x,y)$	1A^0-1a

The map $J_{11}(x, y)$ is due to Halberstadt [Hal98], and is defined by

$$J_{11}(x, y) := \frac{(f_1 f_2 f_3 f_4)^3}{f_5^2 f_6^{11}},$$

where

$$\begin{aligned}
f_1 &:= x^2 + 3x - 6, \\
f_2 &:= 11(x^2 - 5)y + (2x^4 + 23x^3 - 72x^2 - 28x + 127), \\
f_3 &:= 6y + 11x - 19, \\
f_4 &:= 22(x - 2)y + (5x^3 + 17x^2 - 112x + 120), \\
f_5 &:= 11y + (2x^2 + 17x - 34), \\
f_6 &:= (x - 4)y - (5x - 9).
\end{aligned}$$

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