18.310 Homework # 8 **Due Friday, November 7, 2008**

1. A necklace problem:
    a. Suppose that you have a necklace of length 17, and three colors of beads. How many different patterns are there for stringing the beads on the necklace?
    b. Suppose that you use five green beads, five blue beads and 7 yellow beads. How many different patterns are there to string these beads on the necklace?
    c. Suppose that you use three yellow beads, 4 blue beads, and 10 green beads. Now how many dfferent patterns are there to string the beads on the necklace?
2. In the proof of the tree counting theorem, we showed a correspondence between functions from f1; 2; : : : ; n-2g to f1; 2; : : : ; ng and labeled trees. We can use this to prove theorems about these functions.
    a. Let the smallest label of a vertex involved in a cycle be i. In a random function as above, show that the expectation of the ratio of the number of vertices in the cycle containing i and the total number of vertices involved in cycles is larger than one half. (consider for the probability space only functions from f1; 2 ,,, n- 2g to f1; 2; ng which have cycles). (hint: where is the smallest index vertex on the path from n-1 to n on the average?)
    b. Assume that in a random labeled tree, the number of vertices on the path from n - 1 to n is L. What is the expected number of cycles in the corresponding random permutation? Make sure your method is accurate enough to find the constant on the highest-order term.

       Hint: let I_i be an indicator function which is 1 if the i^{th} ranked vertex on this path is closer to n -1 than any of the vertices on the path with labels less than it.

Note: The argument that we used in the tortoise-and-the-hare factoring algorithm to show that the length of the cycle in that algorithm O(pn) can be combined with part (2a) to show that the expected number of vertices in a random tree on the path from n – 1 o n is on the order of the square root of n, This fact with (2a) and (2b) gives a lot of information about the typical structure of random functions, as well as the shape of random labeled trees.

3. Consider the sequence 1, 3, 12, 45, . . . , where $S_n = 3^* S_{n-1} +3^*S_{n-2.}$ Use generating functions to figure out a formula for S_n.
4. Let T_n be the number of ways of tiling a 2 by n strip of squares by tiles which are of size either 1 by 2 or 1 by 4. Give a recurrence which expresses T_n in terms of T_i, with i < n. The generating function for the T's will be rational, and find the roots of the polynomial in the denominator. T_n will then be a sum of multiples of powers of these roots.
    a. On a spreadsheet, compute T_n for n = 1 through 40. Estimate the largest root of the polynomial by taking the ratio T_n=$T_{n\ \ 1}$. How close are you?