

1. Non-Adaptive Weighing

We consider the following classical problem. We have a set of N coins of which exactly one of them is different in weight from the others, all of which are identical. We want to find the identity of the different (bad) coin, by performing as few as possible weighings. Each weighing is to have some of the coins on each side of a balance scale.

Our tasks are to:

1. Determine how many weighings we will need to find the bad coin,
2. Describe a weighing plan for finding it.
3. Create a spreadsheet to locate the bad coin using the plan, given the results of the weighing.

We will assume at first that we have one coin that we know is good in addition to the N doubtful coins.

Our plan is this: we will construct a solution to this problem that states in advance exactly where each coin will be placed (the choice is between being on the left side of the balance, on its right side, or not on it at all,) in each weighing.

Thus we will not use the results of the first weighing to determine what we will do on the next. Finding a scheme like these at first seems harder than merely finding a scheme that is allowed to use information on past results to determine what to do next. In fact it is actually easier to find such a non-looking scheme.

Our plan for finding a “non-adaptive” weighing scheme, which is what we are talking about, is to construct it by induction. First we do the best we can with one weighing, and then show how to find a best construction for k weighings from one for $k-1$ weighings, and that is all we need do.

If we use k weighings for N potentially bad coins and one known good one, our solution can be described by an array or matrix, W , having $N+1$ columns and k rows. The columns will each correspond to a coin (with the 0th column corresponding to the known good coin), and the rows to the weighings. We make $W_{jk} = -1$ if coin k is on the left side of the balance in the j th weighing, $W_{jk} = 1$ if coin k is on its right side in the j th weighing, and $W_{jk} = 0$ if the coin is off the balance in the j th weighing.

What conditions must the matrix W obey?

1. By definition W has the dimensions already given size and entries 1, -1 or 0.
2. The same number of coins must be on either side of the balance in each weighing, or the weighing will tell us nothing. Thus the row sums of W must all be 0.
3. No two of the columns of W both with indices at least 1 can be identical. Otherwise our weighing scheme would not tell us which of the two corresponding coins were bad, if one of them was.

4. No two columns of W both with indices at least 1 can differ only in sign. Otherwise we could not distinguish the heaviness of one of the two corresponding coins from the lightness of the other

Wonderfully enough, if these conditions are obeyed, then we can distinguish the bad coin by examining the outcome of the weighings. We can take the observed results, make a sequence out of it by choosing j th entry 1 if the right side is heavy, -1 if the left is heavy and 0 if the scale balances on the j th weighing and comparing this sequence with each of the columns of W ; the column that the observed sequence matches or is the negative of, is that of the bad coin.

An Upper Bound for the Number of Coins Distinguishable with k Weighings:

There is a standard way to find bounds on how many coins we can handle with k weighings. We compare the number of possible outcomes of the weighings with the number of possible patterns of bad coins that we have to distinguish.

The famous “pigeonhole principle” tells us that we need to have at least as many outcomes as there are patterns, or else some two patterns will have to get the same outcome, and we will not be able to distinguish between them.

With k weighings, the observed sequence will have length k and hence the number of possible outcomes (which can be described by a sequence of length k each entry of which is one of 0, 1 and -1). This means that the number of possible patterns can be at most 3^k .

We have been sneaky and only asked the question, which is the bad coin? There are N possible answers to this question. However, if we ever put a coin on the balance, it being heavy or light will inevitably lead to two different outcomes (In one case its side will go down, in the other it will go up.).

On the other hand, if two of our potentially bad coins do not ever go on the balance we cannot distinguish between them at all. We conclude that all but one of our coins must go on the balance and have its badness lead to two outcomes. We therefore get the following bound:

$$\text{Outcomes} = 3^k \geq 2N - 1 = \text{number of patterns we must distinguish.}$$

For the first few values of k we obtain

k	maximum N value with a good coin and a coin never on the scale
1	2
2	5
3	14
4	41

This maximum value of N is always half of 3^k rounded up to the nearest integer.

What can we accomplish with one weighing?

We use the notation in which 1 means that the indicated coin goes on the left while -1 means it goes on the right, 0 that it stays off scale.

Apart from sign, there is really only one possible arrangement:

$$\begin{array}{r} g \ 1 \ 2 \quad (\text{coin numbers}) \\ 1 \ -1 \ 0 \quad (\text{where the coins go in the weighing}) \end{array}$$

Without a good coin we are dead in the water; we can accomplish nothing at all. The problem doesn't even make sense with only two coins.

With a good coin we can have $N=2$, as in the matrix above, which is the best we can do, a fact that we can deduce from the upper bound just discussed. Notice that coin 2 here does not go on the balance so we will not know if it is heavy or light after this weighing, if it is the bad coin.

Now what can we do with a second weighing?

The following procedure tells us how to add a new weighing to a maximal good weighing scheme with $k-1$ weighings, to obtain a maximally good weighing scheme having k weighings.

What is it?

Suppose the first column, corresponding to the good coin has entries that are all 1, which means that we always put the good coin on the left side of the balance. We can always put the column that is all 0's, which corresponds to the coin that never gets on the balance, on the far right. Then denote what ever else there is between them to be the matrix B .

(When we start with a single weighing, with this definition B is the 1 by 1 matrix consisting of the single entry -1.)

Notice that all row sums in B are -1, since adding the column of 1's of the good coin must make all row sums 0. And also, no column of B is the same or the negative of another because it comes from a "good" weighing scheme.

To create our new scheme we will

1. begin with a column that is all 1's, that represents the good coin.
2. Then we will have a copy of B with 1's underneath them in the last row,
3. then another copy of B with -1's underneath,
4. followed by a copy of $-B$ with 0's underneath.
5. Finally we will add a column that is all zeros except for a -1 in the last place, and

a column that is all 0's.

Let's see what happens by this procedure for $k=2$. B is just (-1).

We get the matrix

$$\begin{array}{cccccc} g & 1 & 2 & 3 & 4 & 5 \\ 1 & -1 & -1 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & -1 & 0 \end{array}$$

Let's now go to three weighings. B now consists of the middle 4 columns above. We get

$$\begin{array}{cccccccccccccccc} g & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\ 1 & -1 & -1 & 1 & 0 & -1 & -1 & 1 & 0 & 1 & 1 & -1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & -1 & 1 & -1 & 0 & -1 & -1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 \end{array}$$

That the row sums work out is little more than the statement: $x + x - x = x$.

Why does this procedure work? Well no column of B is the same or negative of any other, so no column between the first and last can be the same or opposite of any other unless they both come from the same column of B, But if they are the same and in the first two copies of B then they are negatives in the last row, so that they are neither the all the same or all negatives in their entries.. The 0's in the last row corresponding to the -B copy columns make them distinguishable from all columns in the first two copies of B.

And it is easy for you to show that the number of coins obtained this way will be $3^{k/2}$ rounded up.

Notice that you can omit column 13 and change g to obtain a scheme with one fewer coin. There are other similar modifications for two or three fewer coins than the maximum number, but we will leave them for you to figure out in the exercises.

It is obvious that we can obtain a scheme like this for 4 or 5 or more weighings by repeating the process given above. This provides us with a "recursive" definition of our weighing scheme for general k.

By the way, going beyond three weighings is tedious to do by hand, but is quite easy to do on a spreadsheet. Increasing the number of rows of our weighing matrix by 1 involves 4 simple entries: (= -1, 1, -1, 0) and 6 copyings.

An interesting question is: can you come up with an explicit characterization of the columns that appear in this matrix for k rows and a maximum number of columns? This involves determining, given a column, whether it or its negative will appear.

You can indeed figure this out and doing is one of the questions in the problem set.

We write columns above as rows, but here are some answers to random questions of this kind.

Does $0, 1, -1, 0, 0, 1, 1$ occur in the matrix obtained by following this procedure every time a row is added? Answer: NO its negative does.

Does $0, 1, -1, 0, 0, 1, 0$? Answer: YES:

Here is a hint about the answer: suppose the only 0's in a column occur before the first non-zero entry (in B) What is the first non-zero entry of that column?