

# Generating Function Notes

## 18.310, Fall 2005, Prof. Peter Shor

### 1 Counting Change

In this lecture, I'm going to talk about generating functions. We've already seen an example of generating functions. Recall when we had  $n$  independent random variables, each of which was 0 with probability  $p$ , and 1 with probability  $q = 1 - p$ . How did we compute the probability of getting exactly  $k$  1's. What we did was look at the polynomial

$$(p + qx)^n = \binom{n}{0}p^n + \binom{n}{1}p^{n-1}qx + \binom{n}{2}p^{n-2}q^2x^2 + \dots + \binom{n}{n}q^n x^n$$

The coefficient of  $x^k$  was the probability of getting  $k$  1's. This is a simple generating function, and we will generalize this construction in today's lecture. There are a lot more things you can do with generating functions that aren't going to be covered in today's lecture, but it will give you a flavor of how these work. It will also show you how to solve linear recurrence equations, which can be very useful.

Let's look at a related question. Suppose we have 4 pennies, 1 nickel, and 1 dime. For which prices can we give exact change? The answer can be found in a way that's a lot like the probability computation above (in fact, you may have figured out the answer already, but this is an easy example to show you how these generating functions work).

We'll introduce a variable  $x$  so that  $x^n$  will correspond to the ways of making  $n$  cents in change. We can either use 0, 1, 2, 3, or 4 pennies, and 0 or 1 nickels, and 0 or 1 dimes. Thus,

$$(1 + x^{10})(1 + x^5)(1 + x + x^2 + x^3 + x^4)$$

is the generating function which gives the answer. If we use the dime, we use the  $x^{10}$  monomial in the first term. Otherwise, we use the  $x^0 = 1$  monomial, and so forth. Multiplying this out, we get

$$(1 + x^{10})(1 + x^5)(1 + x + x^2 + x^3 + x^4) = 1 + x + x^2 + x^3 + \dots + x^{19},$$

showing that we can obtain any number of cents between 0 and 19.

A little thought will convince you that the coefficient on the  $x^k$  term gives the number of different ways of finding coins that add up to  $k$  cents. We can thus use the same technique to answer a question such as: How many ways can you make

21 cents exact change if you have two dimes, three nickels, and seven pennies? For this problem, the generating function is

$$(1 + x^{10} + x^{20})(1 + x^5 + x^{10} + x^{15})(1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7).$$

The first term is the contribution from the dimes, the second term from the nickels, and the third from the pennies. The coefficient on the  $x^{21}$  term gives you the number of way of making 21 cents exact change. This coefficient is 4, and the terms correspond to coins as follows

$$\begin{array}{l|l} x^{20} \cdot 1 \cdot x & \text{two dimes and one penny} \\ x^{10} \cdot x^{10} \cdot x & \text{one dime, two nickels and one penny} \\ x^{10} \cdot x^5 \cdot x^6 & \text{one dime, one nickel and six pennies} \\ 1 \cdot x^{15} \cdot x^6 & \text{three nickels and six pennies} \end{array}$$

What happens if you have an arbitrary number of dimes, nickels, and pennies? Now, we want to multiply by (for the dimes)

$$1 + x^{10} + x^{20} + x^{30} + x^{40} + x^{50} + \dots$$

where we can have any non-negative integer for the number of dimes. But this infinite series above is just  $1/(1 - x^{10})$ , and it converges as long as  $|x| < 1$ . Thus, the generating function that corresponds to making change using any number of dimes, nickels, and pennies, would be

$$\frac{1}{(1 - x)(1 - x^5)(1 - x^{10})}.$$

The corresponding power series converges if  $|x| \leq 1$ . The essential ingredient we need to know in terms of using it is that it converges in some neighborhood around the  $x = 0$ . Power series that don't necessarily converge anywhere are called *formal power series*; these can still be useful for some purposes, but we won't discuss them any further in this class.

Let's do one more thing with change before we move on. Suppose we are paying with dollar bills and dollar coins. If we count these two forms of money as different, we can make  $n$  dollars in  $n + 1$  different ways: we can use  $k$  coins and  $n - k$  bills for any value of  $k$  between 0 and  $n$ . This corresponds to the identity

$$\frac{1}{(1 - x^{100})^2} = 1 + 2x^{100} + 3x^{200} + 4x^{300} + \dots$$

## 2 Dots and Dashes

Now, let's think about another problem. Suppose that you are sending information using a sequence of two symbols, say dots and dashes, and suppose that the dash is two times as long as a dot<sup>1</sup>. How many different messages can you send in  $n$  time units? Let's call this number  $f_n$ . We'll figure  $f_n$  out for the first few  $n$ . We have

$n$	1	2	3	4	5
$f_n$	1	2	3	5	8
	.	..	...	....	.....
		-	--	---	----
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You may already recognize the pattern: these are Fibonacci numbers. But let's see what we can find out about Fibonacci numbers using generating functions. The recursion for the Fibonacci numbers is

$$f_n = f_{n-1} + f_{n-2}$$

It's not too difficult to see why this works. The last symbol must be either a dot or a dash. If the last symbol is a dash, removing it leaves a sequence two units shorter. If the last symbol is a dot, removing it leaves a sequence one unit shorter. Now, how does this connect to generating functions? Let us define

$$g(x) = \sum_{j=0}^{\infty} f_j x^j.$$

What is  $f_0$ ? It has to be 1, in order to have  $f_2 = f_1 + f_0$ . This makes sense intuitively: there is one message, the empty message, using zero units of time (intuitively, setting  $f_0 = 0$  also makes sense, but it doesn't satisfy the recurrence). What does this recurrence say about  $g(x)$ ? Let's look at the following equations

$$\begin{aligned} g(x) &= 1 + x + 2x^2 + 3x^3 + 5x^4 + 8x^5 + 13x^6 \\ xg(x) &= x + x^2 + 2x^3 + 3x^4 + 5x^5 + 8x^6 \\ x^2g(x) &= x^2 + x^3 + 2x^4 + 3x^5 + 5x^6 \end{aligned}$$

We can see that by multiplying by  $x$  and  $x^2$  we have shifted the terms so that instead of terms  $f_k x^k$ , we get terms  $f_{k-1} x^k$  and  $f_{k-2} x^k$ . We thus get that the

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<sup>1</sup>Real Morse code is a bit more complicated.

equation  $f_k = f_{k-1} + f_{k-2}$  seems to correspond to the (**slightly incorrect**) equation  $g(x) = xg(x) + x^2g(x)$ . Is this the right equation for our generating function  $g(x)$ ? It isn't quite right. All the terms  $x^k$  for  $k > 0$  do cancel in this equation, but it doesn't work for the constant term. To make the constant term correct, we need to add 1 to the right side, obtaining the correct equation

$$g(x) = xg(x) + x^2g(x) + 1.$$

Now, this can be rewritten as

$$g(x) = \frac{1}{1 - x - x^2}.$$

When we have a polynomial in the denominator of a fraction like this, we can factor the polynomial and express it as the sum of two simpler fractions. That is, we first factor the denominator

$$1 - x - x^2 = \left(1 - \left(\frac{1 + \sqrt{5}}{2}\right)x\right) \left(1 - \left(\frac{1 - \sqrt{5}}{2}\right)x\right),$$

We now use the method of partial fractions to rewrite this as

$$g(x) = \frac{a}{1 - \left(\frac{1 + \sqrt{5}}{2}\right)x} + \frac{b}{1 - \left(\frac{1 - \sqrt{5}}{2}\right)x}.$$

Some of you may have seen the method of partial fractions before in calculus when you learned integration. We need to find  $a$  and  $b$ . Elementary algebra gives

$$\begin{aligned} a &= \frac{1}{\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2}\right) \\ b &= -\frac{1}{\sqrt{5}} \left(\frac{1 - \sqrt{5}}{2}\right). \end{aligned}$$

Now, we need to remember what the Taylor series for  $1/(1 - \alpha x)$  is. It is

$$\frac{1}{1 - \alpha x} = 1 + \alpha x + \alpha^2 x^2 + \alpha^3 x^3 + \alpha^4 x^4 + \dots$$

Even those of you who haven't seen Taylor expansions should recognize this as the formula for summing a geometric series.

We thus have that, expanding each of the fractions in the expression for  $g(x)$  above by a Taylor series,

$$\begin{aligned} g(x) &= a \left( 1 + \left(\frac{1+\sqrt{5}}{2}\right) + \left(\frac{1+\sqrt{5}}{2}\right)^2 x^2 + \left(\frac{1+\sqrt{5}}{2}\right)^3 x^3 + \dots \right) \\ &\quad + b \left( 1 + \left(\frac{1-\sqrt{5}}{2}\right) + \left(\frac{1-\sqrt{5}}{2}\right)^2 x^2 + \left(\frac{1-\sqrt{5}}{2}\right)^3 x^3 + \dots \right) \\ &= \frac{1}{\sqrt{5}} \left( \left(\frac{1+\sqrt{5}}{2}\right) + \left(\frac{1+\sqrt{5}}{2}\right)^2 x + \left(\frac{1+\sqrt{5}}{2}\right)^3 x^2 + \left(\frac{1+\sqrt{5}}{2}\right)^4 x^3 + \dots \right) \\ &\quad - \frac{1}{\sqrt{5}} \left( \left(\frac{1-\sqrt{5}}{2}\right) + \left(\frac{1-\sqrt{5}}{2}\right)^2 x + \left(\frac{1-\sqrt{5}}{2}\right)^3 x^2 + \left(\frac{1-\sqrt{5}}{2}\right)^4 x^3 + \dots \right). \end{aligned}$$

Now, this gives us a nice expression for  $f_n$ , the  $n$ th Fibonacci number. We equate the coefficients of  $x^n$  on the left- and right-hand sides of this equation. Since the  $n$ th Fibonacci number  $f_n$  is the coefficient on  $x^n$  in  $g(x)$ , we get

$$f_n = \frac{1}{\sqrt{5}} \left( \left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1} \right).$$

Now, since  $\left|\frac{1-\sqrt{5}}{2}\right| < 1$ , we can see that the second term above is negligible, and  $f_n$  grows as

$$C \left(\frac{1+\sqrt{5}}{2}\right)^n$$

for some  $C$ . (And in fact, it's really easy to figure out  $C$  from this formula.) This means that we have found the asymptotic growth rate for the number of messages that can be encoded by our dots and dashes, and that number of bits sent per time unit is

$$\log_2 \frac{1 + \sqrt{5}}{2}.$$

Now let's generalize this. I'm going to give you a general method for solving linear recurrence equations (also called linear difference equations). If you've taken 18.03, you'll notice that this method looks a lot like the method for solving linear differential equations, and in general, it turns out that recurrence equations often behave a lot like differential equations.

Suppose we have a recurrence equation

$$f_n = \alpha f_{n-1} + \beta f_{n-2} + \gamma f_{n-3}$$

I'm only writing this equation down with three terms, but the generalization to  $k$  terms is obvious, and works just like you'd expect. How do we solve this? What we do is we write down the generating function

$$g(x) = \sum_{j=0}^{\infty} f_j x^j.$$

We then, using the same reasoning we had before, get an equation for  $g(x)$  of the following form:

$$g(x) = \alpha x g(x) + \beta x^2 g(x) + \gamma x^3 g(x) + p(x)$$

where  $p(x)$  is a low-degree polynomial that makes this equation work for the first few elements of the sequence, where the recurrence equation doesn't work (because we don't have a  $f_{-1}$  term). For the Fibonacci number example above, we have  $p(x) = 1$ . Note that if we don't have a  $p(x)$  term, we get the solution  $g(x) = 0$  which, while its coefficients (all 0's) satisfy the linear recurrence equation, doesn't tell us much useful.

We then obtain, as before

$$g(x) = \frac{p(x)}{1 - \alpha x - \beta x^2 - \gamma x^3}$$

Let's suppose we can factor the denominator as follows:

$$1 - \alpha x - \beta x^2 - \gamma x^3 = (1 - r_1 x)(1 - r_2 x)(1 - r_3 x).$$

We then use the method of partial fractions (which you may remember from Calculus) to get

$$g(x) = q(x) + \frac{a}{1 - r_1 x} + \frac{b}{1 - r_2 x} + \frac{c}{1 - r_3 x}$$

where  $a$ ,  $b$  and  $c$  are constants, as before, and where  $q(x)$  is a low degree polynomial that takes care of any anomalies in the first few terms of the sequence (there usually won't be any, so  $q(x)$  usually will be 0). We can then see, by taking a Taylor expansion for this generating function, that a generic term of our sequence will be

$$f_n = ar_1^n + br_2^n + cr_3^n.$$

How did we get the roots  $r_1$ ,  $r_2$  and  $r_3$ ? They are the zeros of the polynomial

$$y^3 - \alpha y^2 - \beta y - \gamma.$$

We can see this by taking  $y = \frac{1}{x}$ , so

$$\begin{aligned} 1 - \alpha x - \beta x^2 - \gamma x^3 &= x^3(y^3 - \alpha y^2 - \beta y - \gamma) \\ &= x^3(y - r_1)(y - r_2)(y - r_3) \\ &= (1 - r_1 x)(1 - r_2 x)(1 - r_3 x) \end{aligned}$$

What happens if we have a double root? I went over this in class, but I'm going to be lazy and not type it up. This case would make a good homework

problem when I teach the class again in the future. You probably did the case of a double root when you took calculus and learned the partial fractions method for integrating fractions with polynomials. If you remember what happened there, or if you remember what happens when you have a double root in differential equations, you can figure out what happens here. When you get  $r$  as a double root, you get a term with

$$(\alpha n + \beta)r^n,$$

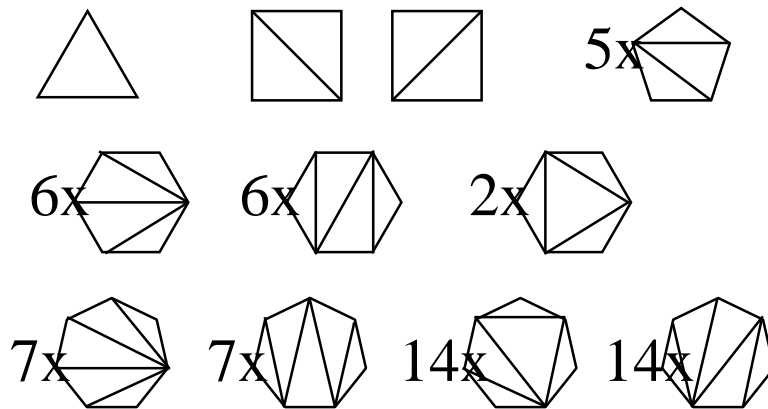
in the expression for  $f_n$ , and if  $r$  appears as a  $k$ -fold root, you get a term  $p(n)r^n$  where  $p(n)$  is a polynomial of degree at most  $k - 1$ .

### 3 A Chord Sequence

Let's count something harder now. (This is the start of Monday's lecture). Let's count how many ways there are of putting chords in a  $k$ -gon to divide it into triangles. We'll call this number  $C_{k-2}$ . This sequence starts as follows:

$k$	3	4	5	6	7
$j = k - 2$	1	2	3	4	5
$C_j$	1	2	5	14	42

as you can see from the following figure. Here I've illustrated one of each essentially way of dividing a  $k$ -gon into triangles, along with the number of times it must be counted (because of symmetry), for  $k \leq 7$ .



How can we find a recurrence for this number? Well, for a  $k$ -gon, let's look at the triangle through the edge  $(k, 1)$ . There must be a third point in this triangle. Let's call it  $j$ . Clearly, we must have  $2 \leq j \leq k - 1$ . If we remove this triangle, we now have two smaller polygons, a  $j$ -sided one, and a  $(k - j + 1)$ -sided one. We now

can divide these polygons up into triangles independently. (We need another figure here.) We thus get that the number of ways of triangulating a  $k$ -gon, given that we have a triangle with vertices  $1, j, k$ , is  $C_{j-2}C_{k-j-1}$ .

One thing we notice is that for this to be true for  $j = 2$  or  $j = k - 1$ , we have to set  $C_0 = 1$ . This takes care of the case where  $j$  is 2 or  $k - 1$ , and one of the two smaller polygons isn't really there (or is just an edge).

Now, we can get a recurrence. Summing over all the  $j$  between 2 and  $k - 1$

$$C_{k-2} = \sum_{j=2}^{k-1} C_{j-2}C_{k-j-1}$$

This formula can use some rethinking of the limits. Let's let  $k' = k - 2$  and  $j' = j - 2$ . We get

$$C_{k'} = \sum_{j'=0}^{k'-1} C_{j'}C_{k'-j'-1}$$

which is a nicer-looking recurrence relation.

The next question is how we evaluate it using generating functions. Let's look at the generating function for counting these triangulations That is,

$$g(x) = \sum_{i=0}^{\infty} C_i x_i = 1 + x + 2x^2 + 5x^3 + 14x^4 + \dots$$

What happens when we square  $g(x)$ . We get

$$\begin{aligned} g(x)^2 &= 1 + (1 + 1)x + (1 \cdot 2 + 1 \cdot 1 + 2 \cdot 1)x^2 + (1 \cdot 5 + 2 \cdot 1 + 1 \cdot 2 + 1 \cdot 5)x^3 + \dots \\ &= \sum_k x^k \sum_{j=0}^k C_j C_{k-j} \end{aligned}$$

You can see that the  $x^k$  term in the expression on the right is the right-hand-side of the recurrence relation we found above for  $C'_k$  (with  $k = k' - 1$ ), so we get

$$g(x)^2 = \sum_{k=0}^{\infty} C_{k+1} x^k.$$

Multiplying this by  $x$  gives a sum with the  $x^j$  coefficient equal to  $C_j x^j$ . So we now have a expression relating  $xg(x)^2$  and  $g(x)$ . We need to make sure we get the smallest terms right. We can check the the constant term is the only one that's wrong, and we can fix that by adding 1 to  $xg(x)^2$ , to get the equation

$$g(x) = 1 + xg(x)^2$$



Now, this is a quadratic equation in  $g(x)$ , so we can use the quadratic formula to solve for it, and we obtain

$$g(x) = \frac{1 \pm \sqrt{1 - 4x}}{2x}.$$

Well, we now have a choice here. Which of the two roots of this equation should we use? We can figure that out by looking at the first term. We should have  $g(0) = 1$ . Depending on which root we choose, we either get  $g(0) = 2/0$  or  $g(0) = 0/0$ . Using l'Hopital's rule, or expanding the power series, we can figure out that in the second case, we indeed have  $g(0) = 1$ , so we get

$$g(x) = \frac{1 - \sqrt{1 - 4x}}{2x}$$

What is this number? Well, to figure that out, we have to expand  $g(x)$  in a power series. We can use the expansion

$$(1-y)^{1/2} = 1 - \frac{1}{2}y + \frac{(1)(-1)}{2^2} \frac{y^2}{2!} - \frac{(1)(-1)(-3)}{2^3} \frac{y^3}{3!} + \frac{(1)(-1)(-3)(-5)}{2^4} \frac{y^4}{4!} - \frac{(1)(-1)(-3)(-5)(-7)}{2^5} \frac{y^5}{5!} + \dots$$

Substituting  $y = 4x$ , this gives

$$g(x) = \frac{1 - \sqrt{1 - 4x}}{2x} = \sum_{k=1}^{\infty} \frac{1}{2x} \frac{1}{2^k} (2k-3)(2k-5)(2k-7) \dots 1 \frac{(4x)^k}{k!}$$

where by  $(2k-3)(2k-5)(2k-7) \dots 1$  we mean the product of odd numbers between 1 and  $2k-3$ , and 1 if  $k=1$ . All the  $-$  signs cancel out, as they should: since we're counting things, we have to get a positive integer.

How do we simplify this expression? Let's equate coefficients on the left-hand- and right-hand-sides of this equation. The coefficient of  $x^j$  on the left-hand-side is  $C_j$ . On the right hand side, we have  $x^{k-1}$ , so we need to take  $k = j + 1$ . Equating coefficients, we get We get that

$$\begin{aligned} C_j &= \frac{1}{2} \frac{1}{2^{j+1}} \frac{(2j-1)(2j-3) \dots 1}{j+1!} 4^{j+1} \\ &= \frac{1}{j+1} \frac{(2j-1)(2j-3) \dots 1}{j!} 2^j \end{aligned}$$

We can now multiply the top and bottom of the above expression by  $j!$  and simplify, to get

$$C_j = \frac{1}{j+1} \frac{(2j)!}{(j!)^2} = \frac{1}{j+1} \binom{2j}{j}$$

Here, we grouped the  $j!$  in the numerator with the  $2^j$  and the product of odd numbers to get  $(2j)!$ . This is the  $j$ 'th Catalan number (in class, I forgot to divide by  $x$ , and told you the wrong result, that it was the  $j-1$ 'th Catalan number).

The Catalan numbers turn up in quite a few places. Prof. Richard Stanley has a section in his enumerative combinatorics book (this section is also on his web page) giving 66 combinatorial interpretations of the Catalan numbers.

## 4 Ups and Downs

The last example we will do is counting the number of alternating permutations. These are permutations that alternate up steps with down steps. For example, the permutation 3,6,1,8,4,7,2,5 is one, since

$$3 < 6 > 1 < 8 > 4 < 7 > 2 < 5$$

Let  $A_n$  be the number of such permutations on the letters  $1, 2, \dots, n$  that start with an up step. It's an easy observation that if we know the number that start with an up step, we also know the number that start with a down step, because these numbers are equal by symmetry. If  $n > 1$ , it's also half the total number, since each alternating permutation (except when  $n = 1$ ) either starts by going up or going down.

How many such permutations are there? We'll go through this example a little more quickly than the last ones, since hopefully you are getting used to this kind of reasoning by now. We can easily count the first few terms of the sequence. This is a good idea in any case in working out this type of problem. You might notice a pattern in these numbers. Even if you don't, in working out the small cases you might get an idea of how to find a recurrence. And in any case, you can use these numbers to check your final results. We have

$$\begin{array}{c|cccccc} k & 0 & 1 & 2 & 3 & 4 & 5 \\ \hline A_k & 1 & 1 & 1 & 2 & 5 & 16 \end{array}$$

We need to find a recurrence on the number of permutations. This means that we somehow need to build each such permutation out of smaller ones. How can we do this?

It's not completely obvious how to find this recurrence. Here's how we do it. What we will do is remove the largest number in the permutation, which will break it into two pieces (one of these pieces may have length 0). Each of these pieces will alternate up and down steps, starting with up. And the first piece will always have odd length, since it has to start with an up step and end with a down step. In our example above, we remove the 8 and get the two sequences 3,6,1 and 4,7,2,5. These aren't examples of alternating permutations, since they aren't on the numbers  $1, 2, \dots, k$  for any  $k$ . However, we can map each of these to a permutation on  $1, 2, \dots, k$  while preserving the relative order of these numbers. For our example

above, these smaller permutations (counted  $A_3$  and  $A_4$  respectively) would be 2,3,1 and 2,4,1,3.

Now, however, we've lost some information about how to construct the original permutation. We need to know which numbers were on the right and which were on the left. That is, we need to know which subset of  $1, 2, 3, \dots, n - 1$  was to the left of  $n$ . In our example above, this subset was  $\{1, 3, 6\}$ . If we know this subset, we can go back and figure out what our original sequence was.

Furthermore, we have found a one-to-one correspondence between, on the one hand, alternating permutations of size  $n$ , and, on the other, a subset of some odd size  $k$  of  $\{1, 2, \dots, n - 1\}$  together with a pair of alternating permutations of size  $k$  and  $n - k - 1$ . This is because if we start with an subset of odd size, and such a pair of alternating permutations, the reconstruction will always give us a length  $n$  alternating permutation. This gives us the recurrence

$$A_n = \sum_{\substack{k=1 \\ k \text{ odd}}}^{2\lfloor n/2 \rfloor - 1} \binom{n-1}{k} A_k A_{n-k-1}$$

since  $\binom{n-1}{k}$  is the number of subsets of size  $k$ .

We can get almost the same recurrence with even  $k$ , by removing the smallest element, 1, instead of the largest. The second alternating permutation (the one to the right of the 1) now starts going down instead of up, but this doesn't matter for our recurrence, as the same number start up and down. This gives the recurrence

$$A_n = \sum_{\substack{k=0 \\ k \text{ even}}}^{2\lfloor (n-1)/2 \rfloor} \binom{n-1}{k} A_k A_{n-k-1}.$$

Adding these two equations together, we get the simpler recurrence

$$2A_n = \sum_{k=0}^{n-1} \binom{n-1}{k} A_k A_{n-k-1}.$$

We now need to find a generating function. Our generating function will be

$$g(x) = \sum_{j=0}^{\infty} A_j \frac{x^j}{j!}.$$

Note that we have stuck in a  $j!$  which we didn't in our previous examples. This is a pretty common thing to do in making generating functions. If you do it, what you get is called an "exponential generating function," and if you don't do it, the technical term for what you get is an "ordinary generating function."

Why did we choose an exponential generating function here? The first reason is the this  $j!$  ensures that the generating function converges, whereas if we tried to write down an ordinary generating function, the function  $A_j$  grows too fast for it to converge. The second reason is that the  $j!$  combines with the  $\binom{n-1}{k}$  term in the recurrence very nicely so that they both nearly cancel. We'll see that now.

If we let  $B_j = A_j/j!$ , our exponential generating function for  $A_j$  becomes an ordinary generating function for  $B_j$ . We have, expanding the term  $\binom{n-1}{k}$  in our recurrence,

$$2A_n = \sum_{k=0}^{n-1} (n-1)! \frac{A_k}{k!} \frac{A_{n-k-1}}{(n-k-1)!}.$$

Now replacing  $A_j$  with  $B_j$  above, we get

$$2n!B_n = \sum_{k=0}^{n-1} (n-1)!B_kB_{n-k-1}, \tag{1}$$

or simplifying,

$$2nB_n = \sum_{k=0}^{n-1} B_kB_{n-k-1}. \tag{2}$$

We know how to handle the right side of this equation using generating functions. We saw this expression when we were looking at the Catalan numbers. This is just the  $x^{n-1}$  term of  $g(x)^2$ . How do we get the left side? Let's remember that

$$g(x) = \sum_j B_j x^j.$$

Differentiating, we get

$$g'(x) = \sum_j jB_j x^{j-1}.$$

Thus, the left side of (4) above is the  $x^{n-1}$  term of  $g'(x)$ . So we now have an equation

$$2g'(x) = g(x)^2 + ?$$

where I've left a '?' in the equation we haven't checked that the coefficients for small powers of  $x$  are equal in this equation, so we may have to add some small polynomial in  $x$  to fix any problems. In fact, this equation doesn't work for the constant term. This is because our recurrence doesn't work for  $A_0$  (this is easy to check).

It's easy to calculate  $B_k$ . The sequence is

$$\begin{array}{c|cccccc} k & 0 & 1 & 2 & 3 & 4 & 5 \\ \hline B_k & 1 & 1 & \frac{1}{2} & \frac{1}{3} & \frac{5}{24} & \frac{2}{15} \end{array}$$

and from these numbers, we can substitute into the differential equation and figure out what the “?” has to be to get it to work. We find

$$2g'(x) = g(x)^2 + 1$$

What’s the solution to this? I don’t know how to figure out the solution to this differential equation by hand, but you can always type it into Maple or Mathematica. If you use Maple, it tells you that the solutions are

$$g(x) = \tan\left(\frac{x}{2} + \theta\right)$$

for some constant  $\theta$ , and it’s easy to check that this works by differentiating. We need to choose  $\theta$  so that  $g(0) = 1$ . It’s easy to see that this means  $\theta = \frac{\pi}{4}$ , so we get

$$g(x) = \tan\left(\frac{x}{2} + \frac{\pi}{4}\right).$$

We can use trigonometric identities to get the equivalent expression

$$g(x) = \tan x + \sec x$$

which is a more useful expression for looking at  $A_k$ . This is because  $\tan$  is an odd function, while  $\sec$  is an even function, so the odd  $A_k$  come from the expansion of  $\tan x$  while the even ones come from the expansion of  $\sec x$ . These are well-studied power series, and the odd  $A_k$  are called *tangent numbers* while the even  $A_k$  are called *secant numbers*.

The generating function for  $g(x)$  can tell us quite a bit about the asymptotic growth rate of the numbers  $A_k$ , and I’ll get to putting that in these notes later. That stuff isn’t urgent, since it won’t help you with the homework.