RESEARCH STATEMENT
Dhruv Ranganathan

My research is in algebraic geometry, with a focus on the development of combinatorial techniques in the study of algebraic varieties, and the application of these techniques to questions of contemporary and classical interest. A recurring motif in my research is the interplay between degenerations of varieties, logarithmic geometry, and the topology of analytic spaces over non-archimedean fields such as $\mathbb{C}(\!(t)\!)$ and $\mathbb{Q}_p$. The combinatorics that emerges from this is part of *tropical* geometry, the systematic study of piecewise-linear shadows of solutions to polynomial equations. This document focuses on the following components of my research in this area.

(1) A conceptual geometric framework for tropical enumerative geometry [A3, A5, A18, A20].
(2) The use of tropical methods to establish new results on the geometry of curves [A13, A16, A17].
(3) The fine and birational geometry of moduli spaces of curves and maps [A2, A19, A21, A22].

I have worked on several other projects, some involving the development of foundations connecting non-archimedean, tropical, and logarithmic geometry [A9, A10, A11], and others of a computational or combinatorial nature [A1, A7, A12, A14, A15], but the following results together illustrate the scope and flavour of my research.

1.1. **Brill–Noether theory, beyond the general curve.** A fact that dates back to Riemann is that every smooth algebraic curve X can be embedded in a projective space. A more refined question asks: *For what r and d does X admit a degree d embedding into $\mathbb{P}^r$?* Brill–Noether theory studies the geometry of the space $W^r_d(X)$ of all degree d embeddings of X into $\mathbb{P}^r$.

For an arbitrary genus g curve, $W^r_d(X)$ can exhibit pathological properties. However, if the complex structure on X is *general*, a celebrated result of Griffiths and Harris [40] shows that

$$\dim W^r_d(X) = \rho(g,r,d) = g - (r+1)(g-d+r).$$

In particular, this determines when such an embedding exists. When the curve is not general, much less is known about the geometry of $W^r_d(X)$. The first measure of the failure of X to be general is its *gonality* – the minimum degree of a non-constant rational function. In work with D. Jensen, I generalize Griffiths and Harris' result to account for this failure of generality [A13].

Theorem 1 (Brill–Noether with a fixed gonality)**.** *Let X be a general curve of gonality k. There exists an explicit numerical invariant $\rho_k(g,r,d)$ such that*

$$\dim W^r_d(X) = \rho_k(g,r,d).$$

Unlike the quantity $\rho(g,r,d)$, the invariant $\rho_k(g,r,d)$ is not predicted by classical theory, but arises as the answer to an analogous combinatorial problem solved by Pflueger [58]. We prove Pflueger's conjecture by blending and extending two highly active but thus far distinct branches of tropical geometry – the logarithmic deformation theory and tropical stable maps methods on the one hand, and chip-firing and tropical Brill–Noether theory on the other, see Theorem 5.

The results and methods open new directions in the subject concerning the Brill–Noether theory of special curves, see Section 5.2.

1.2. **Tropical methods in enumerative geometry.** The genesis of tropical geometry is a remarkable result proved by Mikhalkin in 2003 [54]. Following a proposal of Kontsevich based on mirror symmetry, Mikhalkin proved that the number of *algebraic curves* of given degree and genus in $\mathbb{P}^2$ passing through a given collection of points can be computed by counting piecewise linearly embedded graphs, known as *tropical curves*. This led to elegant combinatorial solutions to fundamental questions in plane enumerative geometry [36, 37].

Tropical methods in enumerative geometry have seen rapid development in the last decade using a range of techniques [31, 32, 55]. However, a precise connection between tropical methods and more traditional degeneration and intersection theoretic techniques has only begun to emerge. The goal of my work in this area has been to unify and generalize these *correspondence principles*, and connect them directly to Gromov–Witten theory by using non-archimedean geometry as a bridge [A5, A6, A18, A19]. In this perspective, tropical moduli spaces furnish *canonical analytic coordinates* on the algebraic moduli spaces. The combinatorics of tropical geometry carry out intersection theory calculations in these canonical coordinates.

This framework has been applied to connect tropical geometry to vertex-operator and Fock space techniques [A3, A4] and plays a central role in the recent applications to Brill–Noether theory discussed above [A13]. The following result, established in work with Len, is the first instance of a tropical enumerative calculation in the presence of non-trivial obstructions [A16]. It generalizes results of Kontsevich [51], Pandharipande [56], and Mikhalkin [54].

Theorem 2 (Curves with fixed j-invariant)**.** *There exists an explicit formula to calculate the number of genus* 1 *curves of fixed j-invariant on any Hirzebruch surface, passing through the expected number of points in general position.*

The proof requires new input from logarithmic deformation theory and the analytic geometry of Artin stacks to overcome a phenomenon known as *superabundance*, which is ubiquitous in potential applications. In future work, I will further develop these methods to recover and generalize results of Ionel, Zinger, and Katz–Qin–Ruan [43, 48, 66, 67] on the enumerative geometry of curves with fixed complex structure.

Much of my current work focuses on the evolution and application of these methods, and their interaction with the theory of virtual fundamental classes.

1.3. **Logarithmic geometry and moduli.** The introduction of the moduli space $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ of stable maps from genus g curves to $\mathbb{P}^r$ was a revolution in enumerative geometry, and led to remarkable progress towards Hilbert's fifteenth problem [39, 51, 52, 57]. Stable maps exhibit their best geometry when $g = 0$ – the space $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$ is smooth and equipped with a combinatorial stratification enabling practical computations. When $g > 0$, the moduli space becomes arbitrarily singular in a precise sense [63], which has left many classical enumerative problems out of reach in higher genus.

In genus 1, famous and difficult work of Vakil and Zinger desingularizes the space $\overline{\mathcal{M}}_{1,n}(\mathbb{P}^r, d)$ by local calculations and explicit blowups. This led to Zinger's celebrated proof of physicists' predictions for the genus 1 enumerative invariants of the quintic threefold [64, 68]. In work with Santos-Parker and Wise, I use insights from tropical geometry to give a simplified proof of Vakil and Zinger's result and establish a number of conceptual generalizations [A21, A22].

By studying combinatorial aspects of the deformation theory of singular curves, we arrive at the concept of *radially aligned genus* 1 *curves* – at first approximation, these are logarithmic curves together with an ordering on the vertices of the tropicalization. This determines a contraction $C \to \overline{C}$ to a curve with an elliptic singularity.

Theorem 3 (Smooth moduli of stable maps in genus 1)**.** *Let Y be a smooth and proper variety. Consider the moduli space $\mathcal{VZ}_{1,n}(Y, \beta)$ for the data $(C \to \overline{C}, f : C \to Y)$, where*

(1) C is an aligned logarithmic curve of genus 1*,*
(2) $f : C \to Y$ is a stable map of degree β factoring through a contraction $C \to \overline{C}$

If $Y = \mathbb{P}^r$, the space $\mathcal{VZ}_{1,n}(\mathbb{P}^r, \beta)$ is smooth and furnishes a modular interpretation for the Vakil–Zinger space. There exist analogous smooth compactifications when Y is an arbitrary toric variety equipped with its logarithmic structure.

In addition to the generalization of Vakil and Zinger's result to toric targets, our work describes their space as a natural moduli space by developing new techniques to understand blowups and main components of moduli spaces through tropical geometry.

2. The tropical inverse problem and applications

The tropicalization of an algebraic variety is a polyhedral complex satisfying certain natural piecewise-linear conditions. The application of tropical techniques to classical geometry often requires an understanding of the inverse problem – *given a "synthetic" tropical variety $\mathscr{P}$, when does there exist an algebraic variety whose tropicalization is $\mathscr{P}$?* New approaches to this question and the resulting applications to geometry have driven much of my recent work [A18, A19, A20, A22].

2.1. **From graphs to Riemann surfaces, via toric stacks.**

Working over a non-archimedean field, the *tropicalization* of a subvariety of a toric variety $Y \subset X$ is the image of Y under a *coordinatewise valuation* map, keeping track of the sizes of solutions to the equations of Y,

$$\mathrm{trop} : X \to \Sigma_X.$$

The inverse question asks *which balanced polyhedral complexes are realized as tropicalizations?* Even for curves, there is no known combinatorial characterization.

A famous result of Speyer provides a partial answer to the question in genus 1 by identifying a sufficient condition for realizability. The desire to generalize Speyer's work has inspired a great deal of research [28, 33, 47, 61, 62], while the lack of sufficient progress on this problem has limited tropical enumerative techniques to low dimensions or genus 0.

The stepping off point of my work in this area is that *all* tropical curves are realizable if one is allowed to replace "toric variety" with "toric stack" [A20].

Theorem 4 (Inversion of tropicalization via toric stacks). *Let $\mathscr{P}$ be an embedded tropical curve. There exists a smooth curve C over a non-archimedean field K and a T-toric variety $X = X(\Delta)$, such that $\mathscr{P}$ is the image of a map of analytic stacks*

$$C^{\mathrm{an}} \to [X^{\mathrm{an}}/T_\circ^{\mathrm{an}}],$$

where $T_\circ^{\mathrm{an}}$ is the non-archimedean analytic compact torus.

This connects the realizability problem for tropical curves to lifting maps from $[X/T]$ to X, which is amenable to cohomological methods. There are numerous immediate applications. For instance, this result implies that limits of realizable curves are realizable, which was invisible to previous approaches to the problem. I also use this to show that Speyer's sufficient condition is not always necessary [A19]. More recently, I have given a full resolution to the tropical realization problem in genus 1, completing Speyer's analysis [A22].

2.2. **Higher genus and applications to linear series.**

Another version of the realizability problem comes from the tropical approach to Brill–Noether theory. Given a curve X over a punctured disk, the stable model of X determines a graph Γ and a deformation retraction

$$\mathrm{trop} : X^{\mathrm{an}} \to \Gamma,$$

from the non-archimedean analytification of the curve. The theory of divisors on graphs identifies divisors of given *degree* and *rank* on Γ, in such a way that divisors on X specialize compatibly to divisors on Γ, see [25, 26, 27]. The lifting question asks: *When does a divisor D on Γ lift to a divisor $\tilde{D}$ on X of the same degree and rank?*

The difficulty of this question is illustrated by a theorem of Cartwright – the realizability problem for divisors of rank 2 on graphs satisfy Mnëv's famous "universality" principle [30]. Unlike the problem for tropical maps, the relevant deformation theoretic techniques are less well-developed.

In work with D. Jensen, I give a new approach to this problem by studying a tropical *map to* $\mathbb{R}^r$ from a divisor on a particular family of graphs. This allows us to deduce lifting theorems for divisors using a new lifting result for maps. We apply this to prove the generalized Brill–Noether theorem (Theorem 1). The main ingredient is a generalization of Speyer's results on tropical realizability to certain tropical curves of arbitrary genus.

Theorem 5 (Speyer's condition in arbitrary genus)**.** *Let Γ be a tropical curve whose stabilization is a chain of cycles. Let*

$$\varphi : \Gamma \to \mathbb{R}^n$$

be a balanced piecewise linear map. Assume that the edge directions of every pair of consecutive cycles of Γ spans $\mathbb{R}^n$. Then there exists an explicit condition determining the realizability of φ in terms of the edge lengths of Γ.

The result relies heavily on the global perspective on realizability involving Artin stacks and logarithmic deformation theory, and seems to be inaccessible by previous techniques [61]. The result leads to a short tropical proof of the Brill–Noether *existence* theorem – classically proved using Schubert calculus by Kempf, Kleiman, and Laksov [49, 50]. No proof without using intersection theory was previously known.

3. Geometry of moduli spaces

The moduli spaces of curves $\mathcal{M}_{g,n}$ and their variations are cornerstones of research in algebraic geometry. Their study has led to the development of spectacular theory, including stacks, logarithmic geometry, enumerative geometry, and the minimal model program. At the same time, their explicit and combinatorial structure has provided concrete but intricate settings in which theory can be tested via computations.

My research in this area has involved the construction of new moduli spaces of embedded curves with good properties [A18, A22], the development of logarithmic techniques for use in the minimal model program for moduli spaces [A21], the study of the combinatorial geometric and topological aspects of compactifications of the moduli space of curves [A2, A7, A8], as well as applications of these techniques to classical questions in geometry [A13, A16].

3.1. Fine geometry of rational curves in toric varieties.

In 2006, Nishinou and Siebert established a far-reaching generalization of Mikhalkin's correspondence theorem, developing a method to tropically count genus 0 curves in any toric variety [55]. In my thesis, I gave a very short conceptual proof of this result by proving a more general geometric statement which showed that much more is true than a numerical correspondence between tropical and algebraic curves: *the geometry of the entire moduli space of logarithmic maps is determined by tropical geometry.*

Let $\mathcal{L}(X)$ be the space of genus 0 logarithmic curves in a toric variety of fixed degree and contact orders with the toric boundary.

Theorem 6 (A geometric correspondence principle)**.** *The space $\mathcal{L}(X)$ is obtained from $\overline{\mathcal{M}}_{0,n} \times X$ by pulling back a subdivision of fans*

$$\Sigma_L \to \mathcal{M}_{0,n}^{\mathrm{trop}} \times X^{\mathrm{trop}},$$

where Σ_L is the moduli space of tropical curves in X^{trop}. Moreover, there is an explicit combinatorial presentation of the operational Chow cohomology ring of $\mathcal{L}(X)$.

This gives one point of access to the global geometry of the space of rational curves in a toric variety, and will be crucial to ongoing and future work concerning a tropical strategy to solve the characteristic numbers problem for toric varieties – a toric generalization of a famous open question from the classical period of enumerative geometry. The understanding of the precise relation between tropical and logarithmic moduli spaces also guided the results of [A21, A22].

3.2. **Birational models of the moduli space of curves.** A fruitful theme of research in the last decade has been to explore the geometric relationship between various compactifications of $\mathcal{M}_{g,n}$, starting with the compactification $\overline{\mathcal{M}}_{g,n}$ by stable curves due to Deligne–Mumford–Knudson. The Hassett–Keel program seeks a modular interpretation for the *canonical model* of $\mathcal{M}_g$ by varying the class of curve singularities allowed [23, 24, 41, 42]. Smyth undertook a beautiful case study of the program program for $\overline{\mathcal{M}}_{1,n}$, showing that for each $m < n$, there are modular compactifications $\overline{\mathcal{M}}_{1,n}(m)$ parametrizing curves with singularities controlled by the discrete parameter m, see [60].

In [A21, Theorem D], Santos-Parker, Wise, and I refine Smyth's study. For each m, there is a birational map,

$$\overline{\mathcal{M}}_{1,n} \dashrightarrow \overline{\mathcal{M}}_{1,n}(m),$$

and it is natural to ask whether this rational map can be factored into geometrically meaningful surgeries. To do this, we introduce *radially aligned* curves of genus 1, by enhancing an ordinary curve with additional tropical data.

Theorem 7. (A universal contraction to m-stable curves) *Let $\overline{\mathcal{M}}_{1,n}^{\mathrm{rad}}$ denote the moduli space of radially aligned n-pointed genus 1 curves. There is a canonical factorization of the rational map $\overline{\mathcal{M}}_{1,n} \dashrightarrow \overline{\mathcal{M}}_{1,n}(m)$ as*

$$\begin{array}{ccc} & \overline{\mathcal{M}}_{1,n}^{\mathrm{rad}} & \\ {}^{\pi}\swarrow & & \searrow^{\tau_m} \\ \overline{\mathcal{M}}_{1,n} & \dashrightarrow & \overline{\mathcal{M}}_{1,n}(m), \end{array}$$

where π is an explicit blowup along boundary strata and τ_m is induced by a contraction of the universal curve.

This result opens the possibility of using logarithmic methods to understand and continue the Hassett–Keel program.

4. Enumerative geometry

4.1. **New frontiers for tropical enumerative geometry.** Since the pioneering work of Mikhalkin and Nishinou–Siebert, tropical methods have been used to prove new results in enumerative geometry, and to provide elegant new proofs of classical ones. Still, there are fundamental difficulties that have not been overcome in the last 10 years and have restricted applications to target dimension 1 and 2, or to genus 0 curves. The primary obstacle has been a satisfactory understanding of the *superabundance* phenomenon exhibited by tropical curves, and the *tropical virtual fundamental class.*

Using the results proved in [A21], non-archimedean methods give rise the first positive genus correspondence theorem without restriction on target dimension.

Theorem 8 (Enumerative geometry in positive genus)**.** *There is a correspondence principle equating the number of elliptic curves in any toric variety incident to subvarieties with a weighted count of tropical elliptic curves in $\mathbb{R}^n$ incident to tropical subvarieties.*

4.2. **Hilbert's fifteenth problem.** The enumeration of curves with contact orders is a famous and difficult problem in algebraic geometry, and was considered by Fulton–Kleiman–MacPherson to be the most important part of Hilbert's fifteenth problem remaining open [35].

In $\mathbb{P}^2$, this question reduces to the characteristic numbers problem, studied by Chasles, Zeuthen, and Schubert in the 1800's: *How many nodal degree d curves of genus g pass through α points in general position and are tangent to β lines in general position?*

Despite heavy interest for over a century, the characteristic numbers problem remains wide open. Partial results are known for curves of low degree or low genus, and with small numbers of tangency conditions. For varieties other than $\mathbb{P}^r$, even less is known – the problem is open for homogeneous varieties in genus 1. Logarithmic and tropical methods provide a new avenue [A17].

Theorem 9 (Characteristic numbers)**.** *Let X be an iterated projective bundle over $\mathbb{P}^r$. There exists a recursive algorithm computing the genus 0 characteristic numbers of X.*

This generalizes work of Pandharipande [57], who computed the numbers for $X = \mathbb{P}^r$, and Bertrand–Brugallé–Mikhalkin [29], who computed the numbers for $X = \mathbb{P}^2$ using tropical methods. With D. Bejleri and J. Wise, I am working to extend the results of [A21, A22] to homogeneous space targets. The methods of [39] are then expected to give concise solutions to the genus 1 characteristic numbers of homogeneous spaces.

5. Future directions

Logarithmic methods have proved to be a powerful tool in modern algebraic geometry, with applications ranging from singularity theory to arithmetic and to mirror symmetry. The theory contains rich combinatorics that has not as yet been fully exploited, but connections between logarithmic geometry and tropical and non-archimedean methods have led to a deeper understanding of this combinatorics, and applications are emerging rapidly. My future work will continue the development and application of logarithmic, non-archimedean, and combinatorial methods to the geometry of curves and their moduli.

5.1. **Logarithmic enumerative geometry.** The two primary tools in Gromov–Witten theory are torus localization and degeneration. The former reduces integrals on moduli spaces to contributions near fixed points of a group action, while the latter studies the contributions near highly singular curves with highly degenerate maps. While they are sometimes interchangeable, the most powerful results in the subject often necessitate their combination.

Logarithmic and tropical methods have provided a sweeping generalization of the conceptual degeneration framework, but a degeneration formula in this generality has not been completed. While a vast program of Abramovich–Chen–Gross–Siebert has been underway for many years, in several geometric situations, including nondegenerate subvarieties of toric varieties, the non-archimedean techniques of [A18] suggest a vastly simplified approach. The interaction of torus localization with tropical methods remain mysterious in general, but are much better understood in the contexts of [A18, A22]. The long term goal of my research in this area is to complete the development of logarithmic degeneration techniques and the interactions with torus localization. As a first step, I will work towards a generalization of a theorem of Maulik and Pandharipande, stating that the logarithmic Gromov–Witten theory of a toric variety over a base scheme reconstructs and can be algorithmically reconstructed from the absolute theory [53]. In parallel, my program will continue to apply these methods to classical questions in enumerative geometry.

5.2. **Geometry of special curves.** Following the determination of closed formulas for the dimensions of $W^r_d(X)$ for a general curve X of fixed gonality [A13], there are a number of natural followup questions concerning the finer geometric aspects of these spaces. It is known that $W^r_d(X)$ is highly singular and non-equidimensional for curves of small gonality. In light of the results of [A21], it is natural to ask if there is a variation of the moduli problem, inspired by tropical geometry, that isolates and desingularizes the various components. This would lead to a generalization of the Gieseker–Petri theorem [38, 44].

Another natural direction comes from the study of Hilbert functions. Intensely studied by Severi and repopularized by Harris, the *maximal rank conjecture* yields a subtle prediction for the Hilbert function of a general curve with a general embedding in projective space. Although the conjecture remains wide open, even partial progress has led to spectacular applications, due to Farkas, Voisin, and many others [34, 65]. For curves of small gonality, no appropriate version of the statement has been formulated. However, the toric and tropical geometry of curves in scrolls, used heavily in the proof of the Brill–Noether theorem for k-gonal curves, reveals a version of the maximal rank conjecture for curves of a fixed gonality. The maximal rank conjecture has already been shown to be amenable to tropical techniques [45, 46].

5.3. **Birational geometry of moduli spaces.** Interactions between logarithmic geometry and alternative compactifications of $\overline{\mathcal{M}}_{g,n}$ remain relatively unexplored. The chief reason is that nodal curves are the unique logarithmically smooth singularities. In [A21], this difficulty is overcome for elliptic singularities by adding data to the tropical moduli problem, opening the possibility for a fruitful interaction between logarithmic geometry and the Hassett–Keel program. Indeed, the methods of loc. cit. appear to be adaptable to logarithmically understand the first critical case in the program – Schubert's pseudostable curves [59].

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