Practice Questions [18:702]

Give an example of a ring with exactly one maximal ideal.

Prove or give a counterexample: the quotient of an ring by a nonzero integral domain by an ideal is also an integral domain.

Prove or give a counterexample: A subring of a noetherian ring is always noetherian.

Describe all maximal ideals in: \( \mathbb{R}[x] \), \( \mathbb{C}[x,y]/(y-x^2) \), \( \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \).

Let \( \mathfrak{p} \) be a prime in \( R \). Prove that \( \mathfrak{p}[x] \) is a prime in \( R[x] \).

Let \( V \subseteq \mathbb{C}^n \) be the set of common zeroes of \( f_1, f_2, \ldots \in \mathbb{C}[x_1, x_2] \). Prove there exists a finite collection of polynomials with the same zero set.

Let \( M, N, P \) be \( R \)-modules. Prove that \( \text{Hom}(M \oplus N, P) \cong \text{Hom}(M, P) \oplus \text{Hom}(N, P) \).

Let \( R \) be a commutative ring and \( M \) an \( R \)-module. Prove that \( R - \text{Hom}_R(M) \) is not necessarily isomorphic to \( M \). Give examples where it is isomorphic to \( M \).

If \( m \) and \( n \) are coprime, prove that \( \mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z} \) is the zero module.