18.702: Problem set 1

Due date: February 16 2018

There are four problems in this set. Please complete them on separate sheets of paper since they will be graded by different people. Be sure to have your name clearly visible. As is standard, you can consult any resource you want and talk to friends, provided you independently understand and write up your solutions. You must list your collaborators and any resources that you consulted outside the textbook.

Policy: A lot of this material is on the internet. If you copy down answers from the internet you’re wasting your time in hilarious fashion, and the graders’ time in infuriating fashion. Don’t be like that. To incentivize honesty, I reserve the right to change your grade if I find that you cannot explain what you have written on your homework.

Throughout this problem set, G will denote a finite group and a representation of G will be a homomorphism ρ : G → GL(W) for a complex vector space W of finite dimension.

0.1 Basics and Definitions

1. Give an example of a nontrivial one-dimensional representation of S_n for all n ≥ 2.

2. A representation ρ : G → GL(W) is called faithful if it has trivial kernel. Prove that every finite group G admits a faithful representation.

3. Let G = Z/n. What are the faithful irreducible representations of G? (See also the problem below).
0.2 Representations of cyclic groups

In class, we sketched a proof that if $G$ is cyclic, then every representation

$$\rho : G \to GL(W)$$

decomposes as a direct sum of 1-dimensional (and hence irreducible) representations. Write a complete proof of this result.

0.3 Representations of abelian groups

We will generalize the result of the previous problem to abelian groups.

1. Begin by proving the following linear algebra fact. Let $A$ and $B$ be diagonalizable linear operators on a vector space $W$, i.e. there exist (possibly different) bases of $W$ in which $A$ and $B$ are each diagonalizable. If $A$ and $B$ commute, prove that they are simultaneously diagonalizable.

2. Look up the classification theorem for finite abelian groups and use the previous part to deduce that if $G$ is abelian, then every representation

$$\rho : G \to GL(W)$$

decomposes as a direct sum of 1-dimensional (and hence irreducible) representations.

0.4 Representations of $S_3$

Let $S_3$ denote the symmetric group on a set of size 3. Let $S_3$ act on $W' = \mathbb{C}^3$ by permuting the coordinates. Let $W$ be the subspace of $W'$ whose coordinates sum to zero. Let $\rho : S_3 \to GL(W)$ be the associated representation obtained by letting $G$ act on the the subspace $W$.

1. Choose a basis of $W$ and write the representation $\rho$ explicitly, i.e. for each of the six elements in $S_3$, write the associated $2 \times 2$ matrix in your favourite basis.

2. Determine (with proof) whether $\rho$ is irreducible or not.

★ Challenge Problem – not to be graded and not worth any credit

You may turn in this problem to me in person in class if you’d like to give it a go.

Let $G$ be a finite group. How many irreducible representations of $G$ are there of dimension one? Your answer can be in terms of $G$, its subgroups, and its quotient groups.