Signatures of Hermitian forms and unitary representations

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Outline

Introduction

Example: $SL(2, \mathbb{R})$

Character formulas

Hermitian forms

Character formulas for invariant forms

Computing easy Hermitian KL polynomials

Unitarity algorithm

Inspirational story
How does symmetry inform mathematics?

Example. \( \int_{-\pi}^{\pi} \sin^5(t)dt = \)？ Zero!

Generalize: \( f = f_{\text{even}} + f_{\text{odd}}, \quad \int_{-a}^{a} f_{\text{odd}}(t)dt = 0. \)

Example. Evolution of initial temp distn of hot ring

\( T(0, \theta) = A + B \cos(m\theta)? \)

\( T(t, \theta) = A + B e^{-c \cdot m^2 t} \cos(m\theta). \)

Generalize: Fourier series expansion of initial temp. . .

Example. \( X \) compact (arithmetic) locally symmetric manifold of dim 128; \( \dim (\, H^{28}(X, \mathbb{C}) \, ) = ? \).

Eight: same as \( H^{28} \) for compact globally symmetric space.

Generalize: \( X = \Gamma \backslash G/K, \, H^p(X, \mathbb{C}) = H^p_{\text{cont}}(G, L^2(\Gamma \backslash G)). \) Decomp \( L^2: \)

\[
L^2(\Gamma \backslash G) = \sum_{\pi \, \text{irr rep of } G} m_\pi(\Gamma) \mathcal{H}_\pi \quad (m_\pi = \text{dim of some aut forms})
\]

Deduce \( H^p(X, \mathbb{C}) = \sum_\pi m_\pi(\Gamma) \cdot H^p_{\text{cont}}(G, \mathcal{H}_\pi). \)

General principal: group \( G \) acts on vector space \( V; \) decompose \( V; \) study pieces separately.
Gelfand’s abstract harmonic analysis

Topological grp $G$ acts on $X$, have questions about $X$.

**Step 1.** Attach to $X$ Hilbert space $\mathcal{H}$ (e.g. $L^2(X)$). Questions about $X \rightsquigarrow$ questions about $\mathcal{H}$.

**Step 2.** Find finest $G$-eqvt decomp $\mathcal{H} = \bigoplus_\alpha \mathcal{H}_\alpha$. Questions about $\mathcal{H} \rightsquigarrow$ questions about each $\mathcal{H}_\alpha$.

Each $\mathcal{H}_\alpha$ is irreducible unitary representation of $G$: indecomposable action of $G$ on a Hilbert space.

**Step 3.** Understand $\hat{G}_u = \text{all irreducible unitary representations of } G$: unitary dual problem.

**Step 4.** Answers about irr reps $\rightsquigarrow$ answers about $X$.

Topic today: **Step 3** for Lie group $G$.

Mackey theory (normal subgps) $\rightsquigarrow$ case $G$ reductive.
What’s a unitary dual look like?

\( G(\mathbb{R}) \) = real points of complex connected reductive alg \( G \)

Problem: find \( \widehat{G(\mathbb{R})}_u \) = irr unitary reps of \( G(\mathbb{R}) \).

Harish-Chandra: \( \widehat{G(\mathbb{R})}_u \subset \widehat{G(\mathbb{R})} = \text{“all” irr reps.} \)

Unitary reps = “all” reps with pos def invt form.

Example: \( G(\mathbb{R}) \) compact \( \Rightarrow \widehat{G(\mathbb{R})}_u = \widehat{G(\mathbb{R})} \) = discrete set.

Example: \( G(\mathbb{R}) = \mathbb{R} \);
\[
\widehat{G(\mathbb{R})} = \left\{ \chi_z(t) = e^{zt} \mid z \in \mathbb{C} \right\} \simeq \mathbb{C}
\]
\[
\widehat{G(\mathbb{R})}_u = \left\{ \chi_i \xi \mid \xi \in \mathbb{R} \right\} \simeq i\mathbb{R}
\]

Suggests: \( \widehat{G(\mathbb{R})}_u \) = real pts of cplx var \( \widehat{G(\mathbb{R})} \). Almost…

\( \widehat{G(\mathbb{R})}_h \) = reps with invt form: \( \widehat{G(\mathbb{R})}_u \subset \widehat{G(\mathbb{R})}_h \subset \widehat{G(\mathbb{R})} \).

Approximately (Knapp): \( \widehat{G(\mathbb{R})} = \text{cplx alg var, real pts} \)
\( \widehat{G(\mathbb{R})}_h \); subset \( \widehat{G(\mathbb{R})}_u \) cut out by real algebraic ineqs.

Today: conjecture making inequalities computable.
Example: $SL(2, \mathbb{R})$ spherical reps

$G(\mathbb{R}) = SL(2, \mathbb{R})$ acts on upper half plane $\mathbb{H}$.

$\sim$ repn $E(\nu)$ on $\nu^2 - 1$ eigenspace of Laplacian $\Delta_{\mathbb{H}}$.

$\nu \in \mathbb{C}$ parametrizes line bdle on circle where bdry values live.

Most $E(\nu)$ irr; always unique irr subrep $J(\nu) \subset E(\nu)$.

Spherical reps for $SL(2, \mathbb{R}) \sim \mathbb{C}/\pm 1$

\[ -i\infty \quad \bullet \quad 1 \quad \bullet \quad -1 \quad \quad i\infty \]

Spectrum of $\Delta_{\mathbb{H}}$ on $L^2(\mathbb{H})$ is $(-\infty, -1]$. Gives unitary reps unitary principal series $\sim \{E(\nu) | \nu \in i\mathbb{R}\}$.

Trivial representations $\sim [\text{const fns on } \mathbb{H}] = J(\pm 1)$.

$J(\nu)$ is Herm. $\Leftrightarrow J(\nu) \sim J(-\nu) \Leftrightarrow \nu \in i\mathbb{R} \cup \mathbb{R}$.

By continuity, signature stays positive from 0 to $\pm 1$.

Complementary series reps $\sim \{E(t) | t \in (-1, 1)\}$. 

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Calculating signatures
Adams et al.
Introduction
$SL(2, \mathbb{R})$
Character formulas
Hermitian forms
Char formulas for invt forms
Herm KL polys
Unitarity algorithm
Inspirational story
The moral[s] of the picture

Spherical unitary dual for $SL(2, \mathbb{R}) \leftrightarrow \mathbb{C}/\pm 1$

$-i\infty \quad 1 \quad i\infty \quad -1$

$SL(2, \mathbb{R}) \quad G(\mathbb{R})$

$E(\nu), \nu \in \mathbb{C}$ \quad $I(\nu), \nu \in a_{\mathbb{C}}^*$

$E(\nu), \nu \in i\mathbb{R}$ \quad $I(\nu), \nu \in i a_{\mathbb{R}}^*$

$J(\nu) \hookrightarrow E(\nu)$ \quad $I(\nu) \rightarrow J(\nu)$

$[-1, 1]$ \quad polytope in $a_{\mathbb{R}}^*$

Will deform Herm forms
unitary axis $ia_{\mathbb{R}}^* \leadsto$
real axis $a_{\mathbb{R}}^*$.

Deformed form pos $\leadsto$
unitary rep.

Reps appear in families, param by $\nu$ in cplx vec space $a^*$.

Pure imag params $\leadsto L^2$ harm analysis $\leadsto$ unitary.

Each rep in family has distinguished irr piece $J(\nu)$.

Difficult unitary reps $\leftrightarrow$ deformation in real param
Signatures for $\text{SL}(2, \mathbb{R})$

Recall $E(\nu) = (\nu^2 - 1)$-eigenspace of $\Delta_{\mathbb{H}}$.

Need “signature” of Herm form on this inf-diml space.

Harish-Chandra (or Fourier) idea:
use $K = \text{SO}(2)$ break into fin-diml subspaces

$$E(\nu)_2 = \{ f \in E(\nu) \mid \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \cdot f = e^{2im\theta} f \}.$$

$$E(\nu) \supset \sum_m E(\nu)_m, \quad \text{(dense subspace)}$$

Decomp is orthogonal for any invariant Herm form.

Signature is $+ \text{ or } -$ for each $m$. For $3 < |\nu| < 5$

$$
\cdots \quad -6 \quad -4 \quad -2 \quad 0 \quad +2 \quad +4 \quad +6 \quad \cdots \\
\cdots \quad + \quad + \quad - \quad + \quad - \quad + \quad + \quad \cdots
$$
Deforming signatures for \( SL(2, \mathbb{R}) \)

Here’s how signatures of the reps \( E(\nu) \) change with \( \nu \).

\[ \nu = 0, \ E(0) \text{ “} \subset \text{” } L^2(\mathbb{H}): \text{ unitary, signature positive.} \]

\[ 0 < \nu < 1, \ E(\nu) \text{ irr: signature remains positive.} \]

\[ \nu = 1, \text{ form finite pos on } J(1) \leftrightarrow SO(2) \text{ rep } 0. \]

\[ \nu = 1, \text{ form has pole, pos residue on } E(1)/J(1). \]

\[ 1 < \nu < 3, \text{ across pole at } \nu = 1, \text{ signature changes.} \]

\[ \nu = 3, \text{ Herm form finite } - + - \text{ on } J(3). \]

\[ \nu = 3, \text{ Herm form has pole, neg residue on } E(3)/J(3). \]

\[ 3 < \nu < 5, \text{ across pole at } \nu = 3, \text{ signature changes. ETC.} \]

Conclude: \( J(\nu) \text{ unitary, } \nu \in [0, 1]; \text{ nonunitary, } \nu \in [1, \infty). \)

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<thead>
<tr>
<th>( \nu )</th>
<th>( \cdots -6 -4 -2 0 +2 +4 +6 \cdots )</th>
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<td>( \nu = 0 )</td>
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\( SO(2) \) reps
From $SL(2, \mathbb{R})$ to reductive $G$

Calculated signatures of invt Herm forms on spherical reps of $SL(2, \mathbb{R})$.

Seek to do “same” for real reductive group. Need.

List of irr reps $= \text{ctble union (cplx vec space)/(fin grp)}$.

reps for purely imag points $\subset L^2(G)$: unitary!

Natural (orth) decomp of any irr (Herm) rep into fin-diml subspaces $\leadsto$ define signature subspace-by-subspace.

Signature at $\nu + i\tau$ by analytic cont $t\nu + i\tau$, $0 \leq t \leq 1$.

Precisely: start w unitary (pos def) signature at $t = 0$; add contribs of sign changes from zeros/poles of odd order in $0 \leq t \leq 1 \leadsto$ signature at $t = 1$. 
Categories of representations

\[ G \text{ cplx reductive alg} \supset G(\mathbb{R}) \text{ real form} \supset K(\mathbb{R}) \text{ max cpt.} \]

Rep theory of \( G(\mathbb{R}) \) modeled on Verma modules . . .

\[ H \subset B \subset G \] maximal torus in Borel subgp,
\[ \mathfrak{h}^* \leftrightarrow \text{highest weight reps} \]
\[ V(\lambda) \quad \text{Verma of hwt } \lambda \in \mathfrak{h}^*, \quad L(\lambda) \quad \text{irr quot} \]

Put cplxification of \( K(\mathbb{R}) = K \subset G \), reductive algebraic.
\( (\mathfrak{g}, K)\)-mod: cplx rep \( V \) of \( \mathfrak{g} \), compatible alg rep of \( K \).

Harish-Chandra: irr \( (\mathfrak{g}, K)\)-mod \[\overset{\sim}{\leftrightarrow}\] “arb rep of \( G(\mathbb{R}) \).”

\[ X \quad \text{parameter set for irr } (\mathfrak{g}, K)\text{-mods} \]
\[ I(x) \quad \text{std } (\mathfrak{g}, K)\text{-mod} \leftrightarrow x \in X \quad J(x) \quad \text{irr quot} \]

Set \( X \) described by Langlands, Knapp-Zuckerman: countable union (subspace of \( \mathfrak{h}^* \))/(subgroup of \( W \)).
Character formulas

Can decompose Verma module into irreducibles

\[ V(\lambda) = \sum_{\mu \leq \lambda} m_{\mu,\lambda} L(\mu) \quad (m_{\mu,\lambda} \in \mathbb{N}) \]

or write a formal character for an irreducible

\[ L(\lambda) = \sum_{\mu \leq \lambda} M_{\mu,\lambda} V(\mu) \quad (M_{\mu,\lambda} \in \mathbb{Z}) \]

Can decompose standard HC module into irreducibles

\[ I(x) = \sum_{y \leq x} m_{y,x} J(y) \quad (m_{y,x} \in \mathbb{N}) \]

or write a formal character for an irreducible

\[ J(x) = \sum_{y \leq x} M_{y,x} I(y) \quad (M_{y,x} \in \mathbb{Z}) \]

Matrices \( m \) and \( M \) upper triang, ones on diag, mutual inverses. Entries are KL polynomials eval at 1:

\[ m_{y,x} = Q_{y,x}(1), \quad M_{y,x} = P_{y,x}(1) \quad (Q_{y,x}, P_{y,x} \in \mathbb{N}[q]). \]
Character formulas for $SL(2, \mathbb{R})$

Recall $(\text{eigenspace of } \Delta_{\mathbb{H}}) = E(\nu) \leftrightarrow J(\nu)$. Prefer dual of $E(\nu) = i_{\text{ev}}(\nu) \rightarrow J(\nu)$.

Need discrete series $l_{\pm}(n)$ ($n = 1, 2, \ldots$) char by

\[
\begin{align*}
    l_{\pm}(n)|_{SO(2)} & = n + 1, \ n + 3, \ n + 5 \ldots \\
    l_{-}(n)|_{SO(2)} & = -n - 1, \ -n - 3, \ -n - 5 \ldots 
\end{align*}
\]

Discrete series reps are irr: $l_{\pm}(n) = J_{\pm}(n)$

Decompose principal series

\[
l_{\text{ev}}(2m + 1) = J_{\text{ev}}(2m + 1) + J_{+}(2m + 1) + J_{-}(2m + 1).
\]

Character formula

\[
J_{\text{ev}}(2m + 1) = i_{\text{ev}}(2m + 1) - i_{+}(2m + 1) - i_{-}(2m + 1).
\]

<table>
<thead>
<tr>
<th>$P_{x,y}$</th>
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Defining Herm dual repn(s)

Suppose \( V \) is a \((g, K)\)-module. Write \( \pi \) for repn map.

Recall Hermitian dual of \( V \)
\[
V^h = \{ \xi : V \to \mathbb{C} \text{ additive} \mid \xi(zv) = \overline{z}\xi(v) \}
\]

Want to construct functor
\[
\text{cplx linear rep } (\pi, V) \rightsquigarrow \text{cplx linear rep } (\pi^h, V^h)
\]
using Hermitian transpose map of operators.

**REQUIRES** twist by conjugate linear automorphism of \( g \).

Assume \( \sigma : G \to G \) antiholom aut, \( \sigma(K) = K \).

Define \((g, K)\)-module \( \pi^h, \sigma \) on \( V^h \),
\[
\pi^h, \sigma(X) \cdot \xi = [\pi(-\sigma(X))]^h \cdot \xi \quad (X \in g, \xi \in V^h).
\]
\[
\pi^h, \sigma(k) \cdot \xi = [\pi(\sigma(k)^{-1})]^h \cdot \xi \quad (k \in K, \xi \in V^h).
\]

Classically \( \sigma_0 \leftrightarrow G(\mathbb{R}) \). We use also \( \sigma_c \leftrightarrow \) compact form of \( G \)
Invariant forms on standard reps

Recall multiplicity formula

\[ I(x) = \sum_{y \leq x} m_{y,x} J(y) \quad (m_{y,x} \in \mathbb{N}) \]

for standard \((g, K)\)-mod \(I(x)\).

Want parallel formulas for \(\sigma\)-invt Hermitian forms.
Need forms on standard modules.

Form on irr \(J(x)\) \xrightarrow{\text{deformation}} Jantzen filt \(I^k(x)\) on std, nondeg forms \(\langle , \rangle^k\) on \(I^k/I^{k+1}\).

Details (proved by Beilinson-Bernstein):

\[ I(x) = I^0 \supset I^1 \supset I^2 \supset \cdots, \quad I^0/I^1 = J(x) \]
\[ I^k/I^{k+1} \text{ completely reducible} \]

\[ [J(y) : I^k/I^{k+1}] = \text{coeff of } q^{(\ell(x) - \ell(y) - k)/2} \text{ in KL poly } Q_{y,x} \]

Hence \(\langle , \rangle_{I(x)} \overset{\text{def}}{=} \sum_k \langle , \rangle^k\), nondeg form on gr \(I(x)\).
Restricts to original form on irr \(J(x)\).
Virtual Hermitian forms

\[ Z = \text{Groth group of vec spaces.} \]
These are mults of irr reps in virtual reps.
\[ Z[X] = \text{Groth grp of finite length reps.} \]

For invariant forms . . .
\[ W = Z \oplus Z = \text{Groth grp of fin diml forms.} \]

Ring structure
\[ (p, q)(p', q') = (pp' + qq', pq' + q'p). \]
Mult of irr-with-forms in virtual-with-forms is in \( W \):
\[ W[X] \approx \text{Groth grp of fin lgth reps with invt forms.} \]

Two problems: invt form \( \langle , \rangle_J \) may not exist for irr \( J \); and \( \langle , \rangle_J \) may not be preferable to \( -\langle , \rangle_J \).
What’s a Jantzen filtration?

$V$ cplx, $\langle , \rangle_t$ Herm forms analytic in $t$, generically nondeg.

$$V = V^0(t) \supset V^1(t) = \text{Rad}(\langle , \rangle_t), \quad J(t) = V^0(t)/V^1(t)$$

$$(p^0(t), q^0(t)) = \text{signature of } \langle , \rangle_t \text{ on } J(t).$$

Question: how does $(p^0(t), q^0(t))$ change with $t$?

First answer: locally constant on open set $V^1(t) = 0$.

Refine answer... define form on $V^1(t)$

$$\langle v, w \rangle^1(t) = \lim_{s \to t} \frac{1}{t - s} < v, w >_s, \quad V_2(t) = \text{Rad}(\langle , \rangle^1(t))$$

$$(p^1(t), q^1(t)) = \text{signature of } \langle , \rangle^1(t).$$

Continuing gives Jantzen filtration

$$V = V^0(t) \supset V^1(t) \supset V^2(t) \cdots \supset V^{m+1}(t) = 0$$

From $t - \epsilon$ to $t + \epsilon$, signature changes on odd levels:

$$p(t + \epsilon) = p(t - \epsilon) + \sum [ -p^{2k+1}(t) + q^{2k+1}(t)].$$
Fix $\sigma$-invt Hermitian form $\langle \cdot, \cdot \rangle_{J(x)}$ on each irr having one; recall Jantzen form $\langle \cdot, \cdot \rangle^n$ on $I(x)^n/I(x)^{n+1}$.

MODULO problem of irrs with no invt form, write

$$(I^n/I^{n+1}, \langle \cdot, \cdot \rangle^n) = \sum_{y \leq x} w_{y,x}(n)(J(y), \langle \cdot, \cdot \rangle_{J(y)})$$

coeffs $w(n) = (p(n), q(n)) \in \mathbb{W}$; summand means

$$p(n)(J(y), \langle \cdot, \cdot \rangle_{J(y)}) \oplus q(n)(J(y), -\langle \cdot, \cdot \rangle_{J(y)})$$

Define Hermitian KL polynomials

$$Q_{y,x}^\sigma = \sum_{n} w_{y,x}(n)q^{(l(x)-l(y)-n)/2} \in \mathbb{W}[q]$$

Eval in $\mathbb{W}$ at $q = 1 \leftrightarrow$ form $\langle \cdot, \cdot \rangle_{I(x)}$ on std.

Reduction to $\mathbb{Z}[q]$ by $\mathbb{W} \rightarrow \mathbb{Z} \leftrightarrow$ KL poly $Q_{y,x}$. 
Hermitian KL polynomials: characters

Matrix $Q^\sigma_{y,x}$ is upper tri, 1s on diag: INVERTIBLE.

$$P^\sigma_{x,y} \overset{\text{def}}{=} (-1)^{l(x)-l(y)}((x, y) \text{ entry of inverse}) \in W[q].$$

Definition of $Q^\sigma_{x,y}$ says

$$\langle \text{gr } I(x), \langle, \rangle_I(x) \rangle = \sum_{y \leq x} Q^\sigma_{x,y}(1)(J(y), \langle, \rangle_J(y));$$

inverting this gives

$$\langle J(x), \langle, \rangle_J(x) \rangle = \sum_{y \leq x} (-1)^{l(x)-l(y)} P^\sigma_{x,y}(1)(\text{gr } I(y), \langle, \rangle_I(y));$$

Next question: how do you compute $P^\sigma_{x,y}$?
Herm KL polys for $\sigma_c$

$\sigma_c = \text{cplx conj for cpt form of } G, \sigma_c(K) = K.$

Plan: study $\sigma_c$-invt forms, relate to $\sigma_0$-invt forms.

Proposition

Suppose $J(x)$ irr $(\mathfrak{g}, K)$-module, real infl char. Then $J(x)$ has
$\sigma_c$-invt Herm form $\langle , \rangle^c_{J(x)}$, characterized by

$\langle , \rangle^c_{J(x)}$ is pos def on the lowest $K$-types of $J(x)$.

Proposition $\implies$ Herm KL polys $Q_{x, y}^{\sigma_c}, P_{x, y}^{\sigma_c}$ well-def.

Coeffs in $\mathcal{W} = \mathbb{Z} \oplus s\mathbb{Z}; s = (0, 1)$ one-diml neg def form.

Conj: $Q_{x, y}^{\sigma_c}(q) = s^{\ell_o(x) - \ell_o(y)}/2 Q_{x, y}(qs), \quad P_{x, y}^{\sigma_c}(q) = s^{\ell_o(x) - \ell_o(y)}/2 P_{x, y}(qs)$.

Equiv: if $J(y)$ occurs at level $k$ of Jantzen filt of $I(x)$, then
Jantzen form is $(-1)^{(l(x) - l(y) - k)/2}$ times $\langle , \rangle_{J(y)}$.

Conjecture is false... but not seriously so. Need an extra power of $s$ on the right side.
Deforming to $\nu = 0$

Have computable **conjectural** formula (omitting $\ell_0$)

$$(J(x), \langle, \rangle^c_{J(x)}) = \sum_{y \leq x} (-1)^{l(x) - l(y)} P_{x,y}(s)(\text{gr } l(y), \langle, \rangle^c_{l(y)})$$

for $\sigma^c$-invt forms in terms of forms on stds, same inf char.

Polys $P_{x,y}$ are KL polys, computed by atlas software.

Std rep $l = l(\nu)$ deps on cont param $\nu$. Put $l(t) = l(t\nu)$, $t \geq 0$.

Apply Jantzen formalism to deform $t$ to 0 . . .

$$\langle, \rangle^c_J = \sum_{l'(0) \text{ std at } \nu' = 0} v_{J,l'} \langle, \rangle^c_{l'(0)} \quad (v_{J,l'} \in W).$$

More rep theory gives formula for $G(\mathbb{R})$-invt forms:

$$\langle, \rangle^c_J = \sum_{l''(0) \text{ std at } \nu' = 0} s^{c(l'')} v_{J,l'} \langle, \rangle^0_{l''(0)}.$$  

$l''(0)$ unitary, so $J$ unitary $\iff$ all coeffs are $(p, 0) \in W$. 
Example of $G_2(\mathbb{R})$

Real parameters for spherical unitary reps of $G_2(\mathbb{R})$

- Unitary rep from $L^2(G)$
- Arthur rep from 6-dim nilp
- Arthur rep from 8-dim nilp
- Arthur rep from 10-dim nilp
- Trivial rep
Possible unitarity algorithm

Hope to get from these ideas a computer program; enter

- real reductive Lie group \( G(\mathbb{R}) \)
- general representation \( \pi \)

and ask whether \( \pi \) is unitary.

Program would say either

- \( \pi \) has no invariant Hermitian form, or
- \( \pi \) has invt Herm form, indef on reps \( \mu_1, \mu_2 \) of \( K \), or
- \( \pi \) is unitary, or
- I’m sorry Dave, I’m afraid I can’t do that.

Answers to finitely many such questions \( \leadsto \) complete description of unitary dual of \( G(\mathbb{R}) \).

This would be a good thing.
An inspirational story

I was an undergrad at University of Chicago, learning interesting math from interesting mathematicians. I left Chicago to work on a Ph.D. with Bert Kostant. After finishing, I came back to Chicago to visit. I climbed up to Paul Sally’s office. Perhaps not all of you know what an interesting mathematician he is. I told him what I’d done in my thesis; since it was representation theory, I hoped he’d find it interesting. He responded kindly and gently, with a question: “What’s it tell you about UNITARY representations?” The answer, regrettably, was, “not much.” So I tried again.