Signatures of Hermitian forms and unitary representations

Jeffrey Adams  Marc van Leeuwen  Peter Trapa
David Vogan  Wai Ling Yee

Representation Theory of Real Reductive Groups,
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Outline

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Character formulas

Hermitian forms

Character formulas for invariant forms

Computing easy Hermitian KL polynomials

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Introduction

\[ G(\mathbb{R}) = \text{real points of complex connected reductive alg } G \]

Problem: find \( \widehat{G(\mathbb{R})}_u = \text{irr unitary reps of } G(\mathbb{R}). \)

Harish-Chandra: \( \widehat{G(\mathbb{R})}_u \subset \widehat{G(\mathbb{R})} = \text{quasisimple irr reps}. \)

Unitary reps = quasisimple reps with pos def invt form.

Example: \( G(\mathbb{R}) \) compact \( \Rightarrow \widehat{G(\mathbb{R})}_u = \widehat{G(\mathbb{R})} = \text{discrete set}. \)

Example: \( G(\mathbb{R}) = \mathbb{R}; \)

\[ \widehat{G(\mathbb{R})} = \{ \chi_z(t) = e^{zt} \ (z \in \mathbb{C}) \} \sim \mathbb{C} \]

\[ \widehat{G(\mathbb{R})}_u = \{ \chi_{i\xi} \ (\xi \in \mathbb{R}) \} \sim i\mathbb{R} \]

Suggests: \( \widehat{G(\mathbb{R})}_u = \text{real pts of cplx var } \widehat{G(\mathbb{R})}. \) Almost . . .

\( \widehat{G(\mathbb{R})}_h = \text{reps with invt form: } \widehat{G(\mathbb{R})}_u \subset \widehat{G(\mathbb{R})}_h \subset \widehat{G(\mathbb{R})}. \)

Approximately (Knapp): \( \widehat{G(\mathbb{R})} = \text{cplx alg var, real pts } \widehat{G(\mathbb{R})}_h; \) subset \( \widehat{G(\mathbb{R})}_u \) cut out by real algebraic ineqs.

Today: conjecture making inequalities computable.
Example: $SL(2, \mathbb{R})$ spherical reps

$G = SL(2, \mathbb{R}) = 2 \times 2$ real matrices of determinant 1

$G$ acts on upper half plane $\mathbb{H} \rightsquigarrow \text{repn } E(\nu)$ on $\nu^2 - 1$ eigenspace of Laplacian $\Delta_{\mathbb{H}}$.

Spectrum of $\Delta_{\mathbb{H}}$ on $L^2(\mathbb{H})$ is $(-\infty, -1] \leftrightarrow \nu \in i\mathbb{R}$.

Most $E(\nu)$ irr; always unique irr subrep $J(\nu) \subset E(\nu)$.

Ex: $E(1) =$ harmonic fns on $\mathbb{H} \supset J(1) =$ constant fns

$J(\nu) \simeq J(\nu') \iff \nu = \pm \nu' \Rightarrow \hat{G}_{sph} = \{J(\nu)\} \simeq \mathbb{C}/\pm 1$.

Cplx conj for real form of $\hat{G}_{sph}$ is $\nu \mapsto -\bar{\nu}$; real points

$$\hat{G}_{sph,h} \simeq (i\mathbb{R} \cup \mathbb{R}) / \pm 1 \subset \mathbb{C}/\pm 1$$

These are sph Herm reps. Unitary pts (Bargmann):

$$\hat{G}_{sph,u} \simeq (i\mathbb{R} \cup [-1, 1]) / \pm 1 \subset \mathbb{C}/\pm 1$$

Moral: have nice families of reps like $E(\nu)$; interesting irreducibles are smaller...
Categories of representations

$G$ cplx reductive alg $\supset G(\mathbb{R})$ real form $\supset K(\mathbb{R})$ max cpt.

Rep theory of $G(\mathbb{R})$ modeled on Verma modules . . .

$H \subset B \subset G$ maximal torus in Borel subgp,

$\mathfrak{h}^* \leftrightarrow$ highest weight reps

$M(\lambda)$ Verma of hwt $\lambda \in \mathfrak{h}^*$, $L(\lambda)$ irr quot

Put cplxification of $K(\mathbb{R}) = K \subset G$, reductive algebraic.

$(\mathfrak{g}, K)$-mod: cplx rep $V$ of $\mathfrak{g}$, compatible alg rep of $K$.

Harish-Chandra: $\text{irr} (\mathfrak{g}, K)$-mod $\leftrightarrow$ “arb rep of $G(\mathbb{R})$.”

$X$ parameter set for irr $(\mathfrak{g}, K)$-mods

$I(x)$ std $(\mathfrak{g}, K)$-mod $\leftrightarrow x \in X$ $J(x)$ irr quot

Set $X$ described by Langlands, Knapp-Zuckerman: countable union (subspace of $\mathfrak{h}^*$)/(subgroup of $W$).
Character formulas

Can decompose Verma module into irreducibles

\[ M(\lambda) = \sum_{\mu \leq \lambda} m_{\mu,\lambda} L(\mu) \quad (m_{\mu,\lambda} \in \mathbb{N}) \]

or write a formal character for an irreducible

\[ L(\lambda) = \sum_{\mu \leq \lambda} M_{\mu,\lambda} M(\mu) \quad (M_{\mu,\lambda} \in \mathbb{Z}) \]

Can decompose standard HC module into irreducibles

\[ I(x) = \sum_{y \leq x} m_{y,x} J(y) \quad (m_{y,x} \in \mathbb{N}) \]

or write a formal character for an irreducible

\[ J(x) = \sum_{y \leq x} M_{y,x} I(y) \quad (M_{y,x} \in \mathbb{Z}) \]

Matrices \( m \) and \( M \) upper triang, ones on diag, mutual inverses. Entries are KL polynomials eval at 1.
Forms and dual spaces

$V$ cplx vec space (or alg rep of $K$, or $(\mathfrak{g}, K)$-mod).

Hermitian dual of $V$

$$V^h = \{ \xi : V \to \mathbb{C} \text{ additive} \mid \xi(zv) = \overline{z}\xi(v) \}$$

(If $V$ is $K$-rep, also require $\xi$ is $K$-finite.)

Sesquilinear pairings between $V$ and $W$

$$\text{Sesq}(V, W) = \{ \langle , \rangle : V \times W \to \mathbb{C}, \text{lin in } V, \text{conj-lin in } W \}$$

$$\text{Sesq}(V, W) \simeq \text{Hom}(V, W^h), \quad \langle v, w \rangle_T = (Tv)(w).$$

Cplx conj of forms is (conj linear) isom

$$\text{Sesq}(V, W) \simeq \text{Sesq}(W, V).$$

Corr (conj linear) isom is Hermitian transpose

$$\text{Hom}(V, W^h) \simeq \text{Hom}(W, V^h), \quad (T^h w)(v) = (Tv)(w).$$

Sesq form $\langle , \rangle_T$ Hermitian if

$$\langle v, v' \rangle_T = \overline{\langle v', v \rangle_T} \iff T^h = T.$$
Defining a rep on $V^h$

Suppose $V$ is a $(\mathfrak{g}, K)$-module. Write $\pi$ for repn map.

Want to construct functor

$$\text{cplx linear rep } (\pi, V) \rightsquigarrow \text{cplx linear rep } (\pi^h, V^h)$$

using Hermitian transpose map of operators. **REQUIRES** twisting by conjugate linear automorphism of $\mathfrak{g}$.

Assume

$$\sigma : G \rightarrow G \text{ antiholom aut, } \sigma(K) = K.$$ 

Define $(\mathfrak{g}, K)$-module $\pi^{h,\sigma}$ on $V^h$,

$$\pi^{h,\sigma}(X) \cdot \xi = [\pi(-\sigma(X))]^h \cdot \xi \quad (X \in \mathfrak{g}, \xi \in V^h).$$

$$\pi^{h,\sigma}(k) \cdot \xi = [\pi(\sigma(k)^{-1})]^h \cdot \xi \quad (k \in K, \xi \in V^h).$$

Traditionally use

$$\sigma_0 = \text{ real form with complexified maximal compact } K.$$ 

We need also

$$\sigma_c = \text{ compact real form of } G \text{ preserving } K.$$
Invariant Hermitian forms

\[ V = (g, K)\)-module, \( \sigma \) antihol aut of \( G \) preserving \( K \).

A \( \sigma \)-invt sesq form on \( V \) is sesq pairing \( \langle , \rangle \) such that

\[
\langle X \cdot v, w \rangle = \langle v, -\sigma(X) \cdot w \rangle, \quad \langle k \cdot v, w \rangle = \langle v, -\sigma(k^{-1}) \cdot w \rangle
\]

\((X \in g; k \in K; v, w \in V)\).

Proposition

\( \sigma \)-invt sesq form on \( V \) \iff \( (g, K) \)-map \( T : V \rightarrow V^{h,\sigma} : \langle v, w \rangle_T = (Tv)(w) \).

Form is Hermitian iff \( T^h = T \).

Assume \( V \) is irreducible.

\( V \cong V^{h,\sigma} \iff \exists \ invt \ sesq \ form \iff \exists \ invt \ Herm \ form \)

A \( \sigma \)-invt Herm form on \( V \) is unique up to real scalar.

\[ T \rightarrow T^h \iff real \ form \ of \ cplx \ line \ Hom_{g,K}(V, V^{h,\sigma}). \]
Invariant forms on standard reps

Recall multiplicity formula
\[ I(x) = \sum_{y \leq x} m_{y,x} J(y) \quad (m_{y,x} \in \mathbb{N}) \]
for standard \((g, K)\)-mod \(I(x)\).

Want parallel formulas for \(\sigma\)-invt Hermitian forms.
Need forms on standard modules.

Form on irr \(J(x)\) \(\xrightarrow{\text{deformation}}\) Jantzen filt \(I_n(x)\) on std, nondeg forms \(\langle , \rangle_n\) on \(I_n/I_{n+1}\).

Details (proved by Beilinson-Bernstein):
\[ I(x) = I_0 \supset I_1 \supset I_2 \supset \cdots, \quad I_0/I_1 = J(x) \]
\(I_n/I_{n+1}\) completely reducible
\[ [J(y): I_n/I_{n+1}] = \text{coeff of } q^{(\ell(x) - \ell(y) - n)/2} \text{ in KL poly } Q_{y,x} \]

Hence \(\langle , \rangle_{I(x)} \overset{\text{def}}{=} \sum_n \langle , \rangle_n\), nondeg form on gr \(I(x)\).
Restricts to original form on irr \(J(x)\).
Virtual Hermitian forms

\[ \mathbb{Z} = \text{Groth group of vec spaces.} \]

These are mults of irr reps in virtual reps.

\[ \mathbb{Z}[X] = \text{Groth grp of finite length reps.} \]

For invariant forms. . .

\[ \mathbb{W} = \mathbb{Z} \oplus \mathbb{Z} = \text{Groth grp of fin diml forms.} \]

Ring structure

\[ (p, q)(p', q') = (pp' + qq', pq' + q'p). \]

Mult of irr-with-forms in virtual-with-forms is in \( \mathbb{W} \):

\[ \mathbb{W}[X] \approx \text{Groth grp of fin lgth reps with invt forms.} \]

Two problems: invt form \( \langle , \rangle_J \) may not exist for irr \( J \); and \( \langle , \rangle_J \) may not be preferable to \( -\langle , \rangle_J \).
Hermitian KL polynomials: multiplicities

Fix $\sigma$-invt Hermitian form $\langle , \rangle_{J(x)}$ on each irr admitting one; recall Jantzen form $\langle , \rangle_n$ on $l(x)_n/l(x)_{n+1}$.

MODULO problem of irrs with no invt form, write

$$(l_n/l_{n-1}, \langle , \rangle_n) = \sum_{y \leq x} w_{y,x}(n)(J(y), \langle , \rangle_{J(y)}),$$

coeffs $w(n) = (p(n), q(n)) \in \mathbb{W}$; summand means

$$p(n)(J(y), \langle , \rangle_{J(y)}) \oplus q(n)(J(y), -\langle , \rangle_{J(y)}).$$

Define Hermitian KL polynomials

$$Q_{y,x}^{\sigma} = \sum_n w_{y,x}(n)q^{(l(x)-l(y)-n)/2} \in \mathbb{W}[q]$$

Eval in $\mathbb{W}$ at $q = 1 \leftrightarrow$ form $\langle , \rangle_{l(x)}$ on std.

Reduction to $\mathbb{Z}[q]$ by $\mathbb{W} \rightarrow \mathbb{Z} \leftrightarrow$ KL poly $Q_{y,x}$. 
Hermitian KL polynomials: characters

Matrix $Q_{y,x}^\sigma$ is upper tri, 1s on diag: INVERTIBLE.

$$P_{x,y}^\sigma \overset{\text{def}}{=} (-1)^{l(x)-l(y)}((x, y) \text{ entry of inverse}) \in \mathbb{W}[q].$$

Definition of $Q_{x,y}^\sigma$ says

$$(\text{gr } I(x), \langle , \rangle_I(x)) = \sum_{y \leq x} Q_{x,y}^\sigma(1)(J(y), \langle , \rangle_J(y));$$

inverting this gives

$$(J(x), \langle , \rangle_J(x)) = \sum_{y \leq x} (-1)^{l(x)-l(y)} P_{x,y}^\sigma(1)(\text{gr } I(y), \langle , \rangle_I(y))$$

Next question: how do you compute $P_{x,y}^\sigma$?
Herm KL polys for $\sigma_c$

$\sigma_c = \text{cplx conj for cpt form of } G, \; \sigma_c(K) = K.$

Plan: study $\sigma_c$-invt forms, relate to $\sigma_0$-invt forms.

Proposition

Suppose $J(x)$ irr $(\mathfrak{g}, K)$-module, real infl char. Then $J(x)$ has $\sigma_c$-invt Herm form $\langle , \rangle_{J(x)}^c$, characterized by

$\langle , \rangle_{J(x)}^c$ is pos def on the lowest $K$-types of $J(x)$.

Proposition $\implies$ Herm KL polys $Q_{x,y}^{\sigma_c}, \; P_{x,y}^{\sigma_c}$ well-def.

Coeffs in $\mathbb{W} = \mathbb{Z} \oplus s\mathbb{Z}; \; s = (0, 1) \leftrightarrow$ one-diml neg def form.

Conj: $Q_{x,y}^{\sigma_c}(q) = s^{\ell_o(x) - \ell_o(y)} \frac{1}{2} Q_{x,y}(qs), \; P_{x,y}^{\sigma_c}(q) = s^{\ell_o(x) - \ell_o(y)} \frac{1}{2} P_{x,y}(qs).

Equiv: if $J(y)$ appears at level $n$ of Jantzen filt of $I(x)$, then Jantzen form is $(-1)^{(l(x) - l(y) - n)/2}$ times $\langle , \rangle_{J(y)}$.

Conjecture is false... but not seriously so. Need an extra power of $s$ (shown in red) on the right side.
Orientation number

Conjecture $\leftrightarrow$ KL polys $\leftrightarrow$ *integral* roots.

Simple form of Conjecture $\Rightarrow$ Jantzen-Zuckerman translation across non-integral root walls preserves signatures of ($\sigma_c$-invariant) Hermitian forms.

It ain’t necessarily so.

$SL(2, \mathbb{R})$: translating spherical principal series from (real non-integral positive) $\nu$ to (negative) $\nu - 2m$ changes sign of form iff $\nu \in (0, 1) + 2\mathbb{Z}$.

**Orientation number** $\ell_o(x)$ is

1. # pairs $(\alpha, -\theta(\alpha))$ cplx nonint, pos on $x$; PLUS
2. # real $\beta$ s.t. $\langle x, \beta^\vee \rangle \in (0, 1) + \epsilon(\beta, x) + 2\mathbb{N}$.

$\epsilon(\beta, x) = 0$ spherical, 1 non-spherical.
Calculating signatures
Adams et al.

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Easy Herm KL polys
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Deforming to \( \nu = 0 \)

Have computable conjectural formula (omitting \( \ell_0 \))

\[
(J(x), \langle \cdot, \cdot \rangle^c_{J(x)}) = \sum_{y \leq x} (-1)^{l(x) - l(y)} P_{x,y}(s)(\text{gr } l(y), \langle \cdot, \cdot \rangle^c_{l(y)})
\]

for \( \sigma^c \)-invt forms in terms of forms on stds, same inf char.

Polys \( P_{x,y} \) are KL polys: computed by atlas.

Std rep \( l = l(\nu) \) deps on cont param \( \nu \). Put \( l(t) = l(t\nu), \ t \geq 0 \).

If std rep \( l = l(\nu) \) has \( \sigma \)-invt form so does \( l(t) \) (\( t \geq 0 \)).

(signature for \( l(t) \)) = (signature on \( l(t + \epsilon) \)), \( \epsilon \geq 0 \) suff small.

Sig on \( l(t) \) differs from \( l(t - \epsilon) \) on odd levels of Jantzen filt:

\[
\langle \cdot, \cdot \rangle_{\text{gr } l(t-\epsilon)} = \langle \cdot, \cdot \rangle_{\text{gr } l(t)} + (s - 1) \sum_m \langle \cdot, \cdot \rangle_{l(t)_{2m+1}/l(t)_{2m+2}}.
\]

Each summand after first on right is known comb of stds, all with cont param strictly smaller than \( t\nu \). ITERATE...

\[
\langle \cdot, \cdot \rangle^c_{J} = \sum_{l'(0) \text{ std at } \nu' = 0} v_{J,l'} \langle \cdot, \cdot \rangle^c_{l'(0)} \quad (v_{J,l'} \in W).
\]
From $\sigma_c$ to $\sigma_0$

Cplx conjs $\sigma_c$ (compact form) and $\sigma_0$ (our real form) differ by Cartan involution $\theta$: $\sigma_0 = \theta \circ \sigma_c$.

Irr $(g, K)$-mod $J \sim J^\theta$ (same space, rep twisted by $\theta$).

Proposition

$J$ admits $\sigma_0$-invt Herm form if and only if $J^\theta \simeq J$. If $T_0: J \sim J^\theta$, and $T_0^2 = \text{Id}$, then

$$\langle v, w \rangle_J^0 = \langle v, T_0 w \rangle_J^c.$$  

$T: J \sim J^\theta \Rightarrow T^2 = z \in \mathbb{C} \Rightarrow T_0 = z^{-1/2} T \sim \sigma$-invt Herm form.

To convert formulas for $\sigma_c$ invt forms $\sim$ formulas for $\sigma_0$-invt forms need intertwining ops $T_J: J \sim J^\theta$, consistent with decomp of std reps.
Equal rank case

\( \text{rk} \, K = \text{rk} \, G \Rightarrow \text{Cartan inv inner: } \exists \tau \in K, \, \text{Ad}(\tau) = \theta. \)

\( \theta^2 = 1 \Rightarrow \tau^2 = \zeta \in Z(G) \cap K. \)

Study reps \( \pi \) with \( \pi(\zeta) = z. \) Fix square root \( z^{1/2}. \)

If \( \zeta \) acts by \( z \) on \( V \), and \( \langle , \rangle^c_V \) is \( \sigma_c \)-invt form, then
\( \langle v, w \rangle^0_V \overset{\text{def}}{=} \langle v, z^{-1/2} \tau \cdot w \rangle^c_V \) is \( \sigma_0 \)-invt form.

\[
\langle , \rangle^c_J = \sum_{l'(0) \text{ std at } \nu' = 0} v_{J, l'} \langle , \rangle^c_{l'(0)} (v_{J, l'} \in W).
\]

translates to
\[
\langle , \rangle^0_J = \sum_{l'(0) \text{ std at } \nu' = 0} v_{J, l'} \langle , \rangle^0_{l'(0)} (v_{J, l'} \in W).
\]

\( l' \) has LKT \( \mu' \Rightarrow \langle , \rangle^0_{l'(0)} \text{ definite, sign } z^{-1/2} \mu(l')(t). \)

J unitary \( \iff \) each summand on right pos def.

Computability of \( v_{J, l'} \) needs conjecture about \( P_{x, y}^{\sigma_c}. \)
General case

Fix “distinguished involution” $\delta_0$ of $G$ inner to $\theta$
Define extended group $G^\Gamma = G \rtimes \{1, \delta_0\}$.
Can arrange $\theta = \text{Ad}(\tau \delta_0)$, some $\tau \in K$.
Define $K^\Gamma = \text{Cent}_{G^\Gamma}(\tau \delta_0) = K \rtimes \{1, \delta_0\}$.

Study $(g, K^\Gamma)$-mods $\leftrightarrow (g, K)$-mods $V$ with $D_0 : V \sim V^{\delta_0}$, $D_0^2 = \text{Id}$.

Beilinson-Bernstein localization: $(g, K^\Gamma)$-mods $\leftrightarrow$ action of $\delta_0$ on $K$-eqvt perverse sheaves on $G/B$.

Should be computable by mild extension of Kazhdan-Lusztig ideas. Not done yet!

Now translate $\sigma_c$-invt forms to $\sigma_0$ invt forms

$$\langle v, w \rangle_V^0 \overset{\text{def}}{=} \langle v, z^{-1/2} \tau \delta_0 \cdot w \rangle_V^c$$

on $(g, K^\Gamma)$-mods as in equal rank case.
Possible unitarity algorithm

Hope to get from these ideas a computer program; enter
  ▶ real reductive Lie group $G(\mathbb{R})$
  ▶ general representation $\pi$
and ask whether $\pi$ is unitary.
Program would say either
  ▶ $\pi$ has no invariant Hermitian form, or
  ▶ $\pi$ has invt Herm form, indef on reps $\mu_1$, $\mu_2$ of $K$, or
  ▶ $\pi$ is unitary, or
  ▶ I’m sorry Dave, I’m afraid I can’t do that.

Answers to finitely many such questions $\leadsto$ complete description of unitary dual of $G(\mathbb{R})$.
This would be a good thing.