Generators of the Unitary Group

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1 What are the generators of U(V)?

Last time we showed that the Symplectic group was generated by the transvections in $SP(V)$ — that is, any element of the group could be written as a product of transvections. Today, we will give a similar result for the Unitary group - showing that it is generated by transvections and quasi-reflections, both of which we will discuss below. In these notes, I will sketch the argument — all the detailed calculations are in Grove's text, and I refer the reader to it for the calculations that I leave out.

For this talk, V will be a unitary space with a Hermitian bilinear form B and a quadratic form Q.

First a brief discussion on transvections.

We define a transvection in $U(V)$ with V as above to be any map that we can, for some scalar a and vector u such that $B(u, u) = 0$, write as

$$\tau_{u,a}v = v + aB(v, u)u \tag{1}$$

For a transvection $\tau_{u,a}$ to be in $U(V)$, it must satisfy

$$B(v, w) = B(\tau v, \tau w) \tag{2}$$

If you plug in the formula for $\tau_{u,a}$, you can find (and Grove does on page 94) that this forces $a = -\overline{a}$. Moreover, recall that u must be isotropic, i.e., $Q(u) = 0$. What this means is that a very constrained subset of all transvections in $GL(V)$ are in $U(V)$, which is not surprising as $U(V)$ is itself a small subset of $GL(V)$. However, we have lost too many transvections, and in fact the subset of $U(V)$ which are transvections will not turn out to be enough to generate $U(V)$.

Let us then look for some more elements of $U(V)$. Well, let's look for maps defined by u, a, where u is not isotropic. Again set $W = u\perp$, and by analogy with the transvections, let us consider a map $\mu \in U(V)$ which is the identity on W. Then $\mu u = au$ for some $a \in F*$ and $Q(u) = Q(\mu u) = a\overline{a}Q(u)$ implies that $a\overline{a} = 1$.

Well, for a nonisotropic u and a which satisfies $a\overline{a} = 1$, define

$$\mu_{u,a}v = v + (a - 1)\frac{B(v, u)}{Q(u)}u \tag{3}$$

You can check that $\mu_{u,a} \in U(V)$, $\mu_{u,a}u = au$ and $\mu_{u,a}$ restricted to W is the identity. So this is what we ordered in the previous paragraph, when looking for a analog to the transvection $\tau_{u,a}$ when u is nonisotropic. Such a map $\mu_{u,a}$ is called a *quasi-reflection along* u by Grove, because for $charF \neq 2$, $\mu_{u,-1}$ is the familiar reflection along u.

2 Another way to write the above generators

Now there is a very clever construction which allows us to treat transvections and quasi-reflections on the same footing. It is a bit complicated to follow, though, so careful:

For any $\sigma \in U(V)$ let $\hat{\sigma} = 1 - \sigma$.

Define the subspace V_σ of V by

$$V_\sigma = \hat{\sigma}V = \{v - \sigma v : v \in V\} \tag{4}$$

$\sigma \in U(V)$ implies that

$$B(\hat{\sigma}x, \hat{\sigma}y) = B(\hat{\sigma}x, y) + B(x, \hat{\sigma}y) \tag{5}$$

for all $x, y \in V$.

One more definition: for $u, v \in V_\sigma$, say $u = \hat{\sigma}x$ and $v = \hat{\sigma}y$, define

$$B_\sigma(u, v) = B_\sigma(\hat{\sigma}u, \hat{\sigma}v) = B(x, \hat{\sigma}y) \tag{6}$$

One can check that B_σ is well defined and is a sesquilinear form on V_σ relative to conjugation. However it is not a Hermitian form because instead of satisfying $B_\sigma(u, v) = -\overline{B_\sigma(v, u)}$, it satisfies

$$B_\sigma(u, v) + \overline{B_\sigma(v, u)} = B(u, v) \tag{7}$$

for all $u, v \in V_\sigma$.

Moreover, B_σ is nondegenerate, because if $B_\sigma(u, v) = 0$ for all $u \in V_\sigma$, then $B(x, \hat{\sigma}y) = 0$ for all $x \in V$. By nondegeneracy of B on V, this implies that $\hat{\sigma}y = 0 = v$ so B_σ is nondegenerate on V_σ.

Take a basis $\{v_1 ... v_r\}$ be a basis for V_σ, and let $\hat{B_\sigma}$ be the matrix of B_σ with respect to this basis, as usual, i.e., $\hat{B_\sigma} = [B_\sigma(v_i, v_j)]$. Let $A = [a_{ij}] = \hat{B_\sigma}^{-1}$. Now we can write σ in terms of B and B_σ (derivation straightforward, in Grove):

$$\sigma x = x - \sum_{i,j} B(x, v_i) a_{ij} v_j \tag{8}$$

3 From a form to the generators of U(V)

So from an arbitrary element $\sigma \in U(V)$ we have constructed a bilinear form B_σ on V_σ. We would like to now go in the opposite direction. The essential thing about the previous section was this form B_σ which was sesquilinear and had this funny property $B_\sigma(u,v) + \overline{B_\sigma(v,u)} = B(u,v)$. Thus we are motivated to consider the following:

Lemma 1. *Suppose that W is a subspace of V and that C is a nondegenerate sesquilinear form on W satisfying $C(u,v) + \overline{C(v,u)} = B(u,v)$ for all $u, v \in W$. Then there is a unique $\sigma \in U(V)$ such that $W = V_\sigma$ and $C = B_\sigma$.*

I omit the proof, which is in Grove on page 96. But such σ will be denoted $\sigma_{W,C}$.

Relative to a basis $\{v_i\}$ for W, we have as above,

$$\sigma_{W,C}x = x - \sum_{i,j} B(x, v_i)a_{ij}v_j \tag{9}$$

for all $x \in V$ and $A = \hat{B_\sigma}^{-1} = \hat{C}^{-1}$

An illustrative example is, suppose that $\dim W = 1$, i.e., $W =< u >$. Then $\sigma_(W,C)(x) = x - aB(x,u)u$ where $a = C(u,u)^{-1}$, and thus by our basic constraint on C, $a^{-1} + \overline{a^{-1}} = Q(u)$. If u is isotropic, that becomes $a + \overline{a} = 0$ and we see that $\sigma_{W,C}$ is just the transvection $\tau_{u,-a}$. On the other hand, if u is anisotropic, then $\sigma_{W,C}$ is the quasi-reflection $\sigma_{u,c}$ with $c = 1 - aQ(u)$.

4 And now the decomposition of an arbitrary element of U(V)

First, another lemma:

Suppose that $W = V_\sigma$ for some $\sigma \in U(V)$ and let W_1 be a nondegenerate B_σ subspace of W. Let $W_2 = W_1 \perp$ in W. Then $W = W_1 \bigoplus W_2$. By our first lemma, there exist σ_1 and $\sigma_2 \in U(V)$ such that $V_{\sigma_i} = W_i$ relative to the sesquilinear forms B_σ restricted to W_i, for $i = 1, 2$.

Lemma 2. *In such a case, $\sigma = \sigma_2\sigma_1$.*

Again, the proof is in Grove, on page 98.

Finally, we will need the fact, also proved in Grove, that if C is a sesquilinear form on W relative to conjugation, then there is a basis $\{w_i\}$ for W with respect to which the matrix of C is upper triangular.

Theorem 3. *If $1 \neq \sigma \in U(V)$ we may factor σ as $\sigma = \sigma_r ... \sigma_1$ with each σ_i either a transvection or a quasi-reflection.*

Proof. Chose a basis $\{w_i\}$ for $W = V_\sigma$ with respect to which B_σ is upper triangular. Then Let σ_1 and σ_2 be as in Lemma 2, with respect to $< w_1 >$, $B_\sigma|_{<w_1>}$ and $< w_2 ... w_r >$, $B_\sigma|_{<w_2...w_r>}$. By Lemma 2, $\sigma = \sigma_2\sigma_1$. As remarked in our illustrative example, σ_1 will be a transvection if w_1 is isotropic, otherwise it will be a quasi-reflection. Finally, the proof follows by induction. □