1 What are the generators of $U(V)$?

Last time we showed that the Symplectic group was generated by the transvections in $SP(V)$ — that is, any element of the group could be written as a product of transvections. Today, we will give a similar result for the Unitary group - showing that it is generated by transvections and quasi-reflections, both of which we will discuss below. In these notes, I will sketch the argument — all the detailed calculations are in Grove’s text, and I refer the reader to it for the calculations that I leave out.

For this talk, $V$ will be a unitary space with a Hermitian bilinear form $B$ and a quadratic form $Q$.

First a brief discussion on transvections.

We define a transvection in $U(V)$ with $V$ as above to be any map that we can, for some scalar $a$ and vector $u$ such that $B(u, u) = 0$, write as

$$\tau_{u,a}v = v + aB(v, u)u$$

For a transvection $\tau_{u,a}$ to be in $U(V)$, it must satisfy

$$B(v, w) = B(\tau v, \tau w)$$

If you plug in the formula for $\tau_{u,a}$, you can find (and Grove does on page 94) that this forces $a = -\overline{a}$. Moreover, recall that $u$ must be isotropic, i.e., $Q(u) = 0$. What this means is that a very constrained subset of all transvections in $GL(V)$ are in $U(V)$, which is not surprising as $U(V)$ is itself a small subset of $GL(V)$. However, we have lost too many transvections, and in fact the subset of $U(V)$ which are transvections will not turn out to be enough to generate $U(V)$.

Let us then look for some more elements of $U(V)$. Well, let’s look for maps defined by $u, a$, where $u$ is not isotropic. Again set $W = u \perp$, and by analogy with the transvections, let us consider a map $\mu \in U(V)$ which is the identity on $W$. Then $\mu u = au$ for some $a \in F^*$ and $Q(\mu u) = Q(\mu u) = a\overline{a}Q(u)$ implies that $a\overline{a} = 1$.

Well, for a nonisotropic $u$ and $a$ which satisfies $a\overline{a} = 1$, define

$$\mu_{u,a}v = v + (a - 1)\frac{B(v, u)}{Q(u)}u$$
2 Another way to write the above generators

Now there is a very clever construction which allows us to treat transvections and quasi-reflections on the same footing. It is a bit complicated to follow, though, so careful:

- For any \( \sigma \in U(V) \) let \( \hat{\sigma} = 1 - \sigma \).
- Define the subspace \( V_\sigma \) of \( V \) by
  \[
  V_\sigma = \hat{\sigma}V = \{ v - \sigma v : v \in V \} \tag{4}
  \]
- \( \sigma \in U(V) \) implies that
  \[
  B(\hat{\sigma}x, \hat{\sigma}y) = B(\hat{\sigma}x, y) + B(x, \hat{\sigma}y) \tag{5}
  \]
  for all \( x, y \in V \).
- One more definition: for \( u, v \in V_\sigma \), say \( u = \hat{\sigma}x \) and \( v = \hat{\sigma}y \), define
  \[
  B_\sigma(u, v) = B_\sigma(\hat{\sigma}u, \hat{\sigma}v) = B(x, \hat{\sigma}y) \tag{6}
  \]
- One can check that \( B_\sigma \) is well defined and is a sesquilinear form on \( V_\sigma \) relative to conjugation. However it is not a Hermitian form because instead of satisfying
  \[
  B_\sigma(u, v) = -\overline{B_\sigma(v, u)} \tag{7}
  \]
  for all \( u, v \in V_\sigma \).
- Moreover, \( B_\sigma \) is nondegenerate, because if \( B_\sigma(u, v) = 0 \) for all \( u \in V_\sigma \), then \( B(x, \hat{\sigma}y) = 0 \) for all \( x \in V \). By nondegeneracy of \( B \) on \( V \), this implies that \( \hat{\sigma}y = 0 = v \) so \( B_\sigma \) is nondegenerate on \( V_\sigma \).
- Take a basis \( \{ v_1 \ldots v_r \} \) be a basis for \( V_\sigma \), and let \( \hat{B}_\sigma \) be the matrix of \( B_\sigma \) with respect to this basis, as usual, i.e., \( \hat{B}_\sigma = [B_\sigma(v_i, v_j)] \). Let \( A = [a_{ij}] = \hat{B}_\sigma^{-1} \).
- Now we can write \( \sigma \) in terms of \( B \) and \( B_\sigma \) (derivation straightforward, in Grove):
  \[
  \sigma x = x - \sum_{i,j} B(x, v_i)a_{ij}v_j \tag{8}
  \]

3 From a form to the generators of \( U(V) \)

So from an arbitrary element \( \sigma \in U(V) \) we have constructed a bilinear form \( B_\sigma \) on \( V_\sigma \). We would like to now go in the opposite direction. The essential thing about the previous section was this form \( B_\sigma \) which was sesquilinear and had
this funny property $B_\sigma(u,v) + \overline{B_\sigma(v,u)} = B(u,v)$. Thus we are motivated to consider the following:

**Lemma 1.** Suppose that $W$ is a subspace of $V$ and that $C$ is a nondegenerate sesquilinear form on $W$ satisfying $C(u,v) + \overline{C(v,u)} = B(u,v)$ for all $u,v \in W$. Then there is a unique $\sigma \in U(V)$ such that $W = V_\sigma$ and $C = B_\sigma$.

I omit the proof, which is in Grove on page 96. But such $\sigma$ will be denoted $\sigma_{W,C}$.

Relative to a basis $\{w_i\}$ for $W$, we have as above,

$$\sigma_{W,C} x = x - \sum_{i,j} B(x, w_i) a_{ij} v_j$$

(9)

for all $x \in V$ and $A = \overline{B_\sigma}^{-1} = \tilde{C}^{-1}$.

An illustrative example is, suppose that $\dim W = 1$, i.e., $W = \langle u \rangle$. Then $\sigma(W,C) = x - aB(x,u)u$ where $a = C(u,u)^{-1}$, and thus by our basic constraint on $C$, $a^{-1} + a^{-1} = Q(u)$. If $u$ is isotropic, that becomes $a + \overline{a} = 0$ and we see that $\sigma_{W,C}$ is just the transvection $\tau_{u,-a}$. On the other hand, if $u$ is anisotropic, then $\sigma_{W,C}$ is the quasi-re
clection $\sigma_{u,c}$ with $c = 1 - aQ(u)$.

### 4 And now the decomposition of an arbitrary element of $U(V)$

First, another lemma:

Suppose that $W = V_\sigma$ for some $\sigma \in U(V)$ and let $W_1$ be a nondegenerate $B_\sigma$ subspace of $W$. Let $W_2 = W_1 \perp W$. Then $W = W_1 \oplus W_2$. By our first lemma, there exist $\sigma_1$ and $\sigma_2 \in U(V)$ such that $V_{\sigma_i} = W_i$ relative to the sesquilinear forms $B_\sigma$ restricted to $W_i$, for $i = 1, 2$.

**Lemma 2.** In such a case, $\sigma = \sigma_2 \sigma_1$.

Again, the proof is in Grove, on page 98.

Finally, we will need the fact, also proved in Grove, that if $C$ is a sesquilinear form on $W$ relative to conjugation, then there is a basis $\{w_i\}$ for $W$ with respect to which the matrix of $C$ is upper triangular.

**Theorem 3.** If $1 \neq \sigma \in U(V)$ we may factor $\sigma$ as $\sigma = \sigma_r \ldots \sigma_1$ with each $\sigma_i$ either a transvection or a quasi-re
clection.

**Proof.** Chose a basis $\{w_i\}$ for $W = V_\sigma$ with respect to which $B_\sigma$ is upper triangular. Then Let $\sigma_1$ and $\sigma_2$ be as in Lemma 2, with respect to $\langle w_1 \rangle$, $B_\sigma \mid \langle w_1 \rangle$ and $\langle w_2 \ldots w_r \rangle$, $B_\sigma \mid \langle w_2 \ldots w_r \rangle$. By Lemma 2, $\sigma = \sigma_2 \sigma_1$. As remarked in our illustrative example, $\sigma_1$ will be a transvection if $w_1$ is isotropic, otherwise it will be a quasi-re
clection. Finally, the proof follows by induction. □