Signatures of Hermitian forms and unitary representations

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Taipei Conference on Representation Theory,
December 20, 2010
Outline

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Introduction

\( G(\mathbb{R}) = \) real points of complex connected reductive alg \( G \)

Problem: find \( \hat{G}(\mathbb{R})_u = \) irr unitary reps of \( G(\mathbb{R}) \).

Harish-Chandra: \( \hat{G}(\mathbb{R})_u \subset \hat{G}(\mathbb{R}) = \) quasisimple irr reps.

Unitary reps = quasisimple reps with pos def invt form.

Example: \( G(\mathbb{R}) \) compact \( \Rightarrow \hat{G}(\mathbb{R})_u = \hat{G}(\mathbb{R}) = \) discrete set.

Example: \( G(\mathbb{R}) = \mathbb{R} \);
\[
\hat{G}(\mathbb{R}) = \{ \chi_z(t) = e^{zt} \ (z \in \mathbb{C}) \} \simeq \mathbb{C}
\]
\[
\hat{G}(\mathbb{R})_u = \{ \chi_{i\xi} \ (\xi \in \mathbb{R}) \} \simeq i\mathbb{R}
\]

Suggests: \( \hat{G}(\mathbb{R})_u = \) real pts of cplx var \( \hat{G}(\mathbb{R}) \). Almost . . .

\( \hat{G}(\mathbb{R})_h = \) reps with invt form: \( \hat{G}(\mathbb{R})_u \subset \hat{G}(\mathbb{R})_h \subset \hat{G}(\mathbb{R}) \).

Approximately (Knapp): \( \hat{G}(\mathbb{R}) = \) cplx alg var, real pts \( \hat{G}(\mathbb{R})_h \); subset \( \hat{G}(\mathbb{R})_u \) cut out by real algebraic ineqs.

Today: algorithm making inequalities computable.
Example: $SL(2, \mathbb{R})$ spherical reps

$G(\mathbb{R}) = SL(2, \mathbb{R})$ acts on upper half plane $\mathbb{H} \sim \text{repn } E(\nu)$ on $\nu^2 - 1$ eigenspace of Laplacian $\Delta_{\mathbb{H}}$.

Unique $SO(2)$-invt eigenfunction $\phi_\nu$ equal 1 at $i$.

Even for $\nu \in i\mathbb{R}$, $E(\nu)$ too fat to carry invt Herm form. Better: $I(\nu) = C^\infty_c(\mathbb{H})/(\text{image of } \Delta_{\mathbb{H}} - (\nu^2 - 1))$.

Have $G$-eqvt linear map $I(\nu) \xrightarrow{A(\nu)} E(\nu)$,

$$A(\nu)f(y) = \int_{\mathbb{H}} f(x)\phi_\nu(x^{-1}y) \, dy.$$

Proposition

For $\nu^2 - 1$ real, $I(\nu)$ admits non-zero invt Herm form

$$\langle f_1, f_2 \rangle = \int_{\mathbb{H}} (A(\nu)f_1(y))\overline{f_2(y)} \, dy$$

radical of form $= \ker A(\nu) = \text{max proper submod of } I(\nu)$.

Define $J(\nu) = I(\nu)/\ker A(\nu)$ (all $\nu \in \mathbb{C}$).
Calculating signatures

Adams et al.

Introduction

Character formulas

Hermitian forms

Char formulas for invt forms

Easy Herm KL polys

Unitarity algorithm

SL(2, ℝ) spherical hermitian dual

\[
I(\nu) = C_c(\mathbb{H})/(\text{im} \Delta_{\mathbb{H}} - (\nu^2 - 1)), \quad J(\nu) = I(\nu)/\ker A(\nu)
\]

\[
J(\nu) \simeq J(\nu') \iff \nu = \pm \nu' \Rightarrow \hat{G}(\mathbb{R})_{sph} = \{J(\nu)\} \simeq \mathbb{C}/\pm 1.
\]

Cplx conj for real form of \( \hat{G}(\mathbb{R})_{sph} \) is \( \nu \mapsto -\nu \); real pts

\[
\hat{G}(\mathbb{R})_{sph,h} \simeq (i\mathbb{R} \cup \mathbb{R})/\pm 1 \subset \mathbb{C}/\pm 1
\]

These are sph Herm reps. Which are unitary?

Need “signature” of Herm form on inf-diml space \( I(\nu) \).

Harish-Chandra idea: \( \hat{K} = SO(2) \leadsto 1\)-diml subspaces

\[
I(\nu)_{2m} = \{ f \in I(\nu) \mid \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \cdot f = e^{2im\theta} f \}.
\]

\[
I(\nu) \supset \sum_m I(\nu)_{2m}, \quad \text{(dense subspace)}
\]

Decomp is orthogonal for any invariant Herm form.

Signature + or − or 0 for each \( m \). Form analytic in \( \nu \), so changes in signature \( \leadsto \) orders of vanishing.
Deforming signatures for $SL(2, \mathbb{R})$

Here’s how signatures of the reps $I(\nu)$ change with $\nu$.

$\nu \in i\mathbb{R}$, $I(\nu)$ “⊂” $L^2(\mathbb{H})$: unitary, signature positive.

$0 < \nu < 1$, $I(\nu)$ irr: signature remains positive.

$\nu = 1$, form pos on quotient $J(1) \leftarrow I(1) \leftrightarrow SO(2)$ rep 0.

$\nu = 1$, form has simple zero, pos “residue” on ker $A(1)$.

$1 < \nu < 3$, across zero at $\nu = 1$, signature changes.

$\nu = 3$, form $- + -$ on $J(3) \leftarrow I(3)$.

$\nu = 3$, form has simple zero, neg “residue” on ker $A(3)$.

$3 < \nu < 5$, across zero at $\nu = 3$, signature changes. ETC.

Conclude: $J(\nu)$ unitary, $\nu \in [0, 1]$; nonunitary, $\nu \in (1, \infty)$.

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Spherical unitary dual for $SL(2, \mathbb{R})$...

...and a preview of more general groups.

Bargmann picture for $SL(2, \mathbb{R})$

\[ \begin{array}{ccc}
-i\infty & 1 & i\infty \\
-1 & & \\
\end{array} \]

\[
SL(2, \mathbb{R}) \quad G(\mathbb{R})
\]

- $l(\nu), \nu \in \mathbb{C}$
- $l(\nu), \nu \in i\mathbb{R}$
- $l(\nu) \rightarrow J(\nu)$
- $[-1, 1]$ polytope in $\mathfrak{a}_\mathbb{R}^*$

Will deform Herm forms
unitary axis $i\mathfrak{a}_\mathbb{R}^* \leftrightarrow$
real axis $\mathfrak{a}_\mathbb{R}^*$.

Deformed form pos $\leftrightarrow$ unitary rep.

Reps appear in families, param by $\nu$ in cplx vec space $\mathfrak{a}^*$.

Pure imag params $\leftrightarrow L^2$ harm analysis $\leftrightarrow$ unitary.

Each rep in family has distinguished irr quotient $J(\nu)$.

Difficult unitary reps $\leftrightarrow$ deformation in real param.
Categories of representations

$G$ cplx reductive alg $\supset G(\mathbb{R})$ real form $\supset K(\mathbb{R})$ max cpt.

Rep theory of $G(\mathbb{R})$ modeled on Verma modules . . .

$H \subset B \subset G$ maximal torus in Borel subgp,

$\mathfrak{h}^* \leftrightarrow$ highest weight reps

$M(\lambda)$ Verma of hwt $\lambda \in \mathfrak{h}^*$, $L(\lambda)$ irr quot

Put cplxification of $K(\mathbb{R}) = K \subset G$, reductive algebraic.

$(\mathfrak{g}, K)$-mod: cplx rep $V$ of $\mathfrak{g}$, compatible alg rep of $K$.

Harish-Chandra: irr $(\mathfrak{g}, K)$-mod $\leftrightarrow$ “arb rep of $G(\mathbb{R})$.”

$X$ parameter set for irr $(\mathfrak{g}, K)$-mods

$I(x)$ std $(\mathfrak{g}, K)$-mod $\leftrightarrow x \in X$ $J(x)$ irr quot

Set $X$ described by Langlands, Knapp-Zuckerman: countable union (subspace of $\mathfrak{h}^*$)/(subgroup of $W$).
Character formulas

Can decompose Verma module into irreducibles

\[ M(\lambda) = \sum_{\mu \leq \lambda} m_{\mu, \lambda} L(\mu) \quad (m_{\mu, \lambda} \in \mathbb{N}) \]

or write a formal character for an irreducible

\[ L(\lambda) = \sum_{\mu \leq \lambda} M_{\mu, \lambda} M(\mu) \quad (M_{\mu, \lambda} \in \mathbb{Z}) \]

Can decompose standard HC module into irreducibles

\[ I(x) = \sum_{y \leq x} m_{y, x} J(y) \quad (m_{y, x} \in \mathbb{N}) \]

or write a formal character for an irreducible

\[ J(x) = \sum_{y \leq x} M_{y, x} I(y) \quad (M_{y, x} \in \mathbb{Z}) \]

Matrices \( m \) and \( M \) upper triang, ones on diag, mutual inverses. Entries are KL polynomials eval at 1.
Forms and dual spaces

$V$ cplx vec space (or alg rep of $K$, or $(g, K)$-mod).

Hermitian dual of $V$

$$V^h = \{ \xi : V \to \mathbb{C} \text{ additive} \mid \xi(zv) = \overline{z}\xi(v) \}$$

(If $V$ is $K$-rep, also require $\xi$ is $K$-finite.)

Sesquilinear pairings between $V$ and $W$

$$\text{Sesq}(V, W) = \{ \langle, \rangle : V \times W \to \mathbb{C}, \text{lin in } V, \text{conj-lin in } W \}$$

$$\text{Sesq}(V, W) \simeq \text{Hom}(V, W^h), \quad \langle v, w \rangle_T = (Tv)(w).$$

Cplx conj of forms is (conj linear) isom

$$\text{Sesq}(V, W) \simeq \text{Sesq}(W, V).$$

Corr (conj linear) isom is Hermitian transpose

$$\text{Hom}(V, W^h) \simeq \text{Hom}(W, V^h), \quad (T^hw)(v) = (Tv)(w).$$

Sesq form $\langle, \rangle_T$ Hermitian if

$$\langle v, v' \rangle_T = \overline{\langle v', v \rangle_T} \iff T^h = T.$$
Defining a rep on $V^h$

Suppose $V$ is a $(g, K)$-module. Write $\pi$ for repn map.

Want to construct functor

$$\text{cplx linear rep } (\pi, V) \leadsto \text{cplx linear rep } (\pi^h, V^h)$$

using Hermitian transpose map of operators. **REQUIRES** twisting by conjugate linear automorphism of $g$.

Assume

$$\sigma : G \to G \text{ antiholom aut}, \quad \sigma(K) = K.$$ 

Define $(g, K)$-module $\pi^{h,\sigma}$ on $V^h$,

$$\pi^{h,\sigma}(X) \cdot \xi = \left[\pi(-\sigma(X))\right]^h \cdot \xi \quad (X \in g, \xi \in V^h).$$

$$\pi^{h,\sigma}(k) \cdot \xi = \left[\pi(\sigma(k)^{-1})\right]^h \cdot \xi \quad (k \in K, \xi \in V^h).$$

Traditionally use

$$\sigma_0 = \text{ real form with complexified maximal compact } K.$$ 

We need also

$$\sigma_c = \text{ compact real form of } G \text{ preserving } K.$$
Invariant Hermitian forms

\[ V = (g, K) \text{-module}, \sigma \text{ antihol aut of } G \text{ preserving } K. \]

A \( \sigma \)-invt sesq form on \( V \) is sesq pairing \( \langle , \rangle \) such that

\[ \langle X \cdot v, w \rangle = \langle v, -\sigma(X) \cdot w \rangle, \quad \langle k \cdot v, w \rangle = \langle v, \sigma(k^{-1}) \cdot w \rangle \]

\[(X \in g; k \in K; v, w \in V).\]

Proposition

\( \sigma \)-invt sesq form on \( V \) \( \iff \) \( (g, K) \)-map \( T : V \to V^{h,\sigma} : \)

\[ \langle v, w \rangle_t = (Tv)(w). \]

Form is Hermitian iff \( T^h = T \).

Assume \( V \) is irreducible.

\[ V \cong V^{h,\sigma} \iff \exists \text{ invt sesq form } \iff \exists \text{ invt Herm form } \]

A \( \sigma \)-invt Herm form on \( V \) is unique up to real scalar.

\[ T \to T^h \iff \text{ real form of cplx line } \text{Hom}_{g,K}(V, V^{h,\sigma}). \]
Invariant forms on standard reps

Recall multiplicity formula

\[ I(x) = \sum_{y \leq x} m_{y,x} J(y) \quad (m_{y,x} \in \mathbb{N}) \]

for standard \((g, K)-\)mod \(I(x)\).

Want parallel formulas for \(\sigma\)-invt Hermitian forms.

Need forms on standard modules.

Form on irr \(J(x)\) \(\xrightarrow{\text{deformation}}\) Jantzen filt \(I_n(x)\) on std, nondeg forms \(\langle \cdot, \cdot \rangle_n\) on \(I_n/I_{n+1}\).

Details (proved by Beilinson-Bernstein):

\[ I(x) = I_0 \supset I_1 \supset I_2 \supset \cdots, \quad I_0/I_1 = J(x) \]

\(I_n/I_{n+1}\) completely reducible

\([J(y): I_n/I_{n+1}] = \text{coeff of } q^{(\ell(x) - \ell(y) - n)/2} \text{ in KL poly } Q_{y,x}\)

Hence \(\langle \cdot, \cdot \rangle_{I(x)} \overset{\text{def}}{=} \sum_n \langle \cdot, \cdot \rangle_n\), nondeg form on gr \(I(x)\).

Restricts to original form on irr \(J(x)\).
Virtual Hermitian forms

\[ Z = \text{Groth group of vec spaces.} \]
These are mults of irr reps in virtual reps.
\[ Z[X] = \text{Groth grp of finite length reps.} \]

For invariant forms...
\[ W = Z \oplus Z = \text{Groth grp of fin diml forms.} \]

Ring structure
\[(p, q)(p', q') = (pp' + qq', pq' + q'p).\]

Mult of irr-with-forms in virtual-with-forms is in \( W \):
\[ W[X] \approx \text{Groth grp of fin lgth reps with invt forms.} \]

Two problems: invt form \( \langle \cdot, \cdot \rangle_J \) may not exist for irr \( J \);
and \( \langle \cdot, \cdot \rangle_J \) may not be preferable to \(-\langle \cdot, \cdot \rangle_J\).
Hermitian KL polynomials: multiplicities

Fix $\sigma$-invt Hermitian form $\langle , \rangle_{J(x)}$ on each irr admitting one; recall Jantzen form $\langle , \rangle_n$ on $I(x)_n/I(x)_{n+1}$. MODULO problem of irrs with no invt form, write

$$(I_n/I_{n-1}, \langle , \rangle_n) = \sum_{y \leq x} w_{y,x}(n)(J(y), \langle , \rangle_{J(y)}),$$

coeffs $w(n) = (p(n), q(n)) \in \mathbb{W}$; summand means

$$p(n)(J(y), \langle , \rangle_{J(y)}) \oplus q(n)(J(y), -\langle , \rangle_{J(y)}).$$

Define Hermitian KL polynomials

$$Q_{y,x}^\sigma = \sum_n w_{y,x}(n)q^{(I(x)-I(y)-n)/2} \in \mathbb{W}[q]$$

Eval in $\mathbb{W}$ at $q = 1 \iff$ form $\langle , \rangle_{I(x)}$ on std.

Reduction to $\mathbb{Z}[q]$ by $\mathbb{W} \to \mathbb{Z} \iff$ KL poly $Q_{y,x}$. 
Hermitian KL polynomials: characters

Matrix $Q_{y,x}^\sigma$ is upper tri, 1s on diag: INVERTIBLE.

$$P_{x,y}^\sigma \overset{\text{def}}{=} (-1)^{l(x)-l(y)}((x, y) \text{ entry of inverse}) \in \mathbb{W}[q].$$

Definition of $Q_{x,y}^\sigma$ says

$$(\text{gr } l(x), \langle \cdot , \cdot \rangle_{l(x)}) = \sum_{y \leq x} Q_{x,y}^\sigma(1)(J(y), \langle \cdot , \cdot \rangle_{J(y)});$$

inverting this gives

$$(J(x), \langle \cdot , \cdot \rangle_{J(x)}) = \sum_{y \leq x} (-1)^{l(x)-l(y)} P_{x,y}^\sigma(1)(\text{gr } l(y), \langle \cdot , \cdot \rangle_{l(y)});$$

Next question: how do you compute $P_{x,y}^\sigma$?
Herm KL polys for $\sigma_c$

$\sigma_c = \text{cplx conj for cpt form of } G, \sigma_c(K) = K.$

Plan: study $\sigma_c$-invt forms, relate to $\sigma_0$-invt forms.

**Proposition**

Suppose $J(x)$ irr $(\mathfrak{g}, K)$-module, real infl char. Then $J(x)$ has $\sigma_c$-invt Herm form $\langle \cdot, \cdot \rangle^c_{J(x)}$, characterized by $\langle \cdot, \cdot \rangle^c_{J(x)}$ is pos def on the lowest $K$-types of $J(x)$.

**Proposition** $\implies$ Herm KL polys $Q^\sigma_{x,y}, P^\sigma_{x,y}$ well-def.

Coefs in $\mathbb{W} = \mathbb{Z} \oplus s\mathbb{Z}; s = (0, 1) \leftrightarrow \text{one-diml neg def form.}$

Conj: $Q^\sigma_{x,y}(q) = s \frac{\ell_o(x) - \ell_o(y)}{2} Q_{x,y}(qs), \quad P^\sigma_{x,y}(q) = s \frac{\ell_o(x) - \ell_o(y)}{2} P_{x,y}(qs)$.  

Equiv: if $J(y)$ appears at level $n$ of Jantzen filt of $I(x)$, then Jantzen form is $(-1)^{(l(x)-l(y)-n)/2}$ times $\langle \cdot, \cdot \rangle_{J(y)}$.

Conjecture is false... but not seriously so. Need an extra power of $s$ on the right side.
Orientation number

Conjecture $\leftrightarrow$ KL polys $\leftrightarrow$ *integral* roots.

Simple form of Conjecture $\Rightarrow$ Jantzen-Zuckerman translation across non-integral root walls preserves signatures of ($\sigma_c$-invariant) Hermitian forms.

It ain’t necessarily so.

$SL(2, \mathbb{R})$: translating spherical principal series from (real non-integral positive) $\nu$ to (negative) $\nu - 2m$ changes sign of form iff $\nu \in (0, 1) + 2\mathbb{Z}$.

*Orientation number* $\ell_o(x)$ is

1. # pairs $(\alpha, -\theta(\alpha))$ cplx nonint, pos on $x$; PLUS
2. # real $\beta$ s.t. $\langle x, \beta^\vee \rangle \in (0, 1) + \epsilon(\beta, x) + 2\mathbb{N}$. 

$\epsilon(\beta, x) = 0$ spherical, 1 non-spherical.
Deforming to $\nu = 0$

Have computable formula (omitting $\ell_o$)

$$\langle J(x), \langle , \rangle^c_{J(x)} \rangle = \sum_{y \leq x} (-1)^{l(x) - l(y)} P_{x,y}(s)(\text{gr } l(y), \langle , \rangle^c_{l(y)})$$

for $\sigma^c$-invt forms in terms of forms on stds, same inf char.

Polys $P_{x,y}$ are KL polys, computed by atlas software.

Std rep $l = l(\nu)$ deps on cont param $\nu$. Put $l(t) = l(t\nu)$, $t \geq 0$.

If std rep $l = l(\nu)$ has $\sigma$-invt form so does $l(t)$ ($t \geq 0$).

(signature for $l(t)$) = (signature on $l(t + \epsilon)$), $\epsilon \geq 0$ suff small.

Sig on $l(t)$ differs from $l(t - \epsilon)$ on odd levels of Jantzen filt:

$$\langle , \rangle_{\text{gr } l(t - \epsilon)} = \langle , \rangle_{\text{gr } l(t)} + (s - 1) \sum_m \langle , \rangle_{l(t)_{2m+1}/l(t)_{2m+2}}.$$

Each summand after first on right is known comb of stds, all with cont param strictly smaller than $t\nu$. ITERATE...

$$\langle , \rangle^c_{J} = \sum_{\text{std at } \nu' = 0} v_{J,\nu'} \langle , \rangle^c_{l'_{(0)}} \quad (v_{J,\nu'} \in \mathbb{W}).$$
From $\sigma_c$ to $\sigma_0$

Cplx conjs $\sigma_c$ (compact form) and $\sigma_0$ (our real form) differ by Cartan involution $\theta$: $\sigma_0 = \theta \circ \sigma_c$.

$Irr (g, K)$-mod $J \sim J^\theta$ (same space, rep twisted by $\theta$).

Proposition

$J$ admits $\sigma_0$-invt Herm form if and only if $J^\theta \subset J$. If $T_0: J \sim J^\theta$, and $T_0^2 = \text{Id}$, then

$$\langle v, w \rangle^0_J = \langle v, T_0 w \rangle^c_J.$$ 

$T: J \sim J^\theta \Rightarrow T^2 = z \in \mathbb{C} \Rightarrow T_0 = z^{-1/2} T \sim \sigma$-invt Herm form.

To convert formulas for $\sigma_c$ invt forms $\sim$ formulas for $\sigma_0$-invt forms need intertwining ops $T_J: J \sim J^\theta$, consistent with decomp of std reps.
Equal rank case

\[ \text{rk } K = \text{rk } G \Rightarrow \text{Cartan inv inner: } \exists \tau \in K, \text{Ad}(\tau) = \theta. \]

\[ \theta^2 = 1 \Rightarrow \tau^2 = \zeta \in Z(G) \cap K. \]

Study reps \( \pi \) with \( \pi(\zeta) = z \). Fix square root \( z^{1/2} \).

If \( \zeta \) acts by \( z \) on \( V \), and \( \langle , \rangle^c_V \) is \( \sigma_c \)-invt form, then
\[ \langle v, w \rangle^0_J \overset{\text{def}}{=} \langle v, z^{-1/2} \tau \cdot w \rangle^c_V \]

is \( \sigma_0 \)-invt form.

\[ \langle , \rangle^c_J = \sum_{l'(0) \text{ std at } \nu' = 0} v_{J,l'} \langle , \rangle^c_{l'(0)} \quad (v_{J,l'} \in W). \]

translates to
\[ \langle , \rangle^0_J = \sum_{l'(0) \text{ std at } \nu' = 0} v_{J,l'} \langle , \rangle^0_{l'(0)} \quad (v_{J,l'} \in W). \]

\( l' \) has LKT \( \mu' \Rightarrow \langle , \rangle^0_{l'(0)} \) \text{ definite, sign } z^{-1/2} \mu'(\tau). \)

J unitary \( \Leftrightarrow \) each summand on right pos def.
General case

Fix “distinguished involution” $\delta_0$ of $G$ inner to $\theta$
Define extended group $G^\Gamma = G \rtimes \{1, \delta_0\}$.
Can arrange $\theta = \text{Ad}(\tau \delta_0)$, some $\tau \in K$.
Define $K^\Gamma = \text{Cent}_{G^\Gamma}(\tau \delta_0) = K \rtimes \{1, \delta_0\}$.
Study $(g, K^\Gamma)$-mods $\leftrightarrow (g, K)$-mods $V$ with
$D_0 : V \xrightarrow{\sim} V^{\delta_0}$, $D_0^2 = \text{Id}$.

Beilinson-Bernstein localization: $(g, K^\Gamma)$-mods $\leftrightarrow$ action of $\delta_0$ on $K$-eqvt perverse sheaves on $G/B$.

Should be computable by mild extension of Kazhdan-Lusztig ideas. Not done yet!

Now translate $\sigma_c$-invt forms to $\sigma_0$ invt forms

$$\langle v, w \rangle^0_V \overset{\text{def}}{=} \langle v, z^{-1/2} \tau \delta_0 \cdot w \rangle^c_V$$

on $(g, K^\Gamma)$-mods as in equal rank case.
Possible unitarity algorithm

Hope to get from these ideas a computer program; enter

- real reductive Lie group $G(\mathbb{R})$
- general representation $\pi$

and ask whether $\pi$ is unitary.

Program would say either

- $\pi$ has no invariant Hermitian form, or
- $\pi$ has invt Herm form, indef on reps $\mu_1, \mu_2$ of $K$, or
- $\pi$ is unitary, or
- I’m sorry Dave, I’m afraid I can’t do that.

Answers to finitely many such questions $\leadsto$ complete description of unitary dual of $G(\mathbb{R})$.

This would be a good thing.