Generators of the Symplectic Group

Yaim Cooper

May 11, 2005

1 Basic Concepts

In this talk we consider only vector spaces V of positive even dimension.

Definition. *A nondegenerate alternate form B on V is called a symplectic form on V.*

Note: It is the desire for the symplectic form to be nondegenerate which forces V to have even dimension.

Definition. *A hyperbolic pair in V is an ordered pair (u, v) of vectors (necessarily linearly independent) with $B(u, v) = 1$.*

Definition. *An ordered set $\{u_1, v_1 ... u_m, v_m\}$ is a symplectic basis for V.*

With respect to a symplectic basis, then, the matrix of B is $\begin{pmatrix} 0 & 1 & 0 & 0 & \dots \\ -1 & 0 & 1 & 0 & \\ 0 & -1 & 0 & 0 & \\ \vdots & & & & \ddots \end{pmatrix}$.

Definition. *An invertible linear transformation τ of V is said to be symplectic if $B(\tau a, \tau b) = B(a, b)$.*

The set of all such linear maps is closed under composition and thus forms a subgroup of $GL(V)$ called the symplectic group on V and denoted $Sp(V)$. As we've seen already in similar situations, $Sp(V)$ does not depend which form we chose on V because every symplectic form is equivalent on V. However, if we do fix a basis (equivalent to choosing a form) we see that $B(\tau a, \tau b) = B(a, b)$ is equivalent to the matrix equation $T^t BT = B$ where T is the matrix of τ with respect to the chosen basis.

2 Transvections

The best way to picture a transvection is as a shear - you fix some hyperplane, and take the orthogonal vector to that hyperplane and apply a shear, so that it makes an angle $\theta \neq 90°$ with the hyperplane. But enough of pictures, let us give a real definition.

Definition. *A linear map $\tau : V \to V$ is a transvection with fixed hyperplane W if $\tau|_W = Id_W$ and $\tau v - v \in W$ for all $v \in V$.*

Given a vector u and a scalar a, we can construct a transvection $\tau_{u,a}$ by $\tau_{u,a}(v) = v + aB(v,u)u$. This is a transvection because it fixes the hyperplane $u^\perp$ and $\tau_{u,a}(v) - v = aB(v,u)u$ is a scalar multiple of u, which is in $u^\perp$. In fact, every transvection can be expressed in this way for some $u \in V$ and some scalar a. This is shown in *Grove*.

3 The Main Result

Today's main result is that $Sp(V)$ is generated by symplectic transvections. The proof will be done in three parts.

Proof. Denote by T the group generated by all transvections τ. We will show that $T = Sp(V)$.

Part (a): T acts transitively on $V \backslash \{0\}$.

Take $v \neq w \in V\backslash\{0\}$.

Case 1. $B(v,w) \neq 0$. Set $a = \frac{1}{B(v,w)}$ and $u = v - w$. Then $\tau_{u,a}(v) = v + aB(v,u)u = v + \frac{B(v,v-w)}{B(v,w)}(v-w) = v - (v-w) = w$.

Case 2. $B(v,w) = 0$. We want a vector z such that $B(v,z) \neq 0 \neq B(w,z)$. This is possible because in our basis of hyperbolic vectors, $v = \sum_1^n a_i v_i + b_i u_i$ and $w = \sum_1^n c_i v_i + d_i u_i$. If for some i, a_i or $b_i \neq 0$ and c_i or $d_i \neq 0$, some linear combination of v_i and u_i satisfies our requirement for z. Otherwise, for all i if a_i or $b_i \neq 0$ then c_i *and* $d_i = 0$ and vice versa. Since neither v nor w is zero, there is a nonzero coefficient α_i of v, where α can be a or b and nonzero γ_j with γ as c or d, and $i \neq j$. Then we can take z to be an appropriate nonzero linear combination of v_i and u_i plus an appropriate nonzero linear combination of v_j and u_j.

Having constructed the desired vector z, we can now use Case 1 to construct τ_1 and τ_2 such that $\tau_1(v) = z$, $\tau_2(z) = w$ and thus $\tau_2\tau_1(v) = w$.

Part (b): T is transitive on hyperbolic pairs.

We want to show that there is a product of transvections which maps (u_1, v_1) to (u_2, v_2). By part (a), there is a transvection τ mapping u_1 to u_2. Thus $\tau : (u_1, v_1) \to (u_2, \tau(v_1))$. Let us denote $\tau(v_1$ by v_3 We want $\sigma \in T$ such that $\sigma(u_2) = u_2$ and $\sigma(v_3) = v_2$.

Note 1. *Note that if $B(\alpha,\beta) \neq 0$, if we set $\gamma = \alpha - \beta$ and $a_\gamma = \frac{1}{B(\alpha,\beta)}$, then if $B(u_2,\gamma) = 0$, then by Part (a), $\tau_{\gamma,a_\gamma}(u_2) = u_2$ and $\tau_{\gamma,a_\gamma}(\alpha) = \beta$.*

Case 1. $B(v_3, v_2) \neq 0$.

Then let $\gamma = v_3 - v_2$. By Note 1, $\tau_{\gamma,a_\gamma}(u_2) = u_2$ and $\tau_{\gamma,a_\gamma}(v_3) = v_2$, as desired.

Case 2. $B(v_3, v_2) = 0$

In this case, $B(v_3, u_2 + v_3) = -1$. Also, $B(u_2 + v_3, v_2) = B(u_2, v_2) \neq 0$ Moreover, $B(u_2, -u_2) = 0 = B(u_2, u_2 + v_3 - v_2$ So, by Note 1, we can construct

σ_1 and σ_2 respectively such that both fix u_2 and $sigma_1(v_3) = u_2 + v_3$ and $sigma_2(u_2 + v_3) = v_2$. Thus $\sigma_2\sigma_1\tau \in T$ and $\sigma_2\sigma_1\tau(u_1, v_1) = (u_2, v_2)$, as desired.
Part (c) The symplectic group $Sp(V)$ is generated by symplectic transvections.

We prove the result by induction on m, where $2m = n = dimV$. The case $m = 1$ is follows from the fact that for $m = 1$, $Sp(V) = Sl(V)$and the fact that $Sl(V)$ is generated by transvections.

The inductive step: Chose a hyperbolic pair (u, v) in V and let W be the hyperbolic plane they span. Then $V = W \bigoplus W^{\perp}$. Take any $\sigma \in Sp(V)$. Then $(\sigma u, \sigma v)$ is a hyperbolic pair, and by Part (b), there exists $\tau \in T$ with $\tau\sigma u = u$ and $\tau\sigma v = v$, so $\tau\sigma|_W = Id_W$. Moreover, $\tau\sigma|_{W^\perp} \in Sp(W^\perp)$. By induction, $\tau\sigma|_{W^\perp}$ is a product of symplectic transvections on $W^\perp$. Since any transvection on $(W^\perp)$ can be extended to a transvection on V which includes W in the fixed hyperplane, we see that in V, $\tau\sigma \in T$ and hence $\sigma \in T$.

$\square$