Maximal parabolic subgroups of symplectic groups

These notes are intended as an outline for a long presentation to be made early in April. They describe certain particularly interesting subgroups of a symplectic group, and some of the related geometry of symplectic group actions. My hope is that some of you will make parallel presentations for each of the other families of classical groups.

I am going to follow fairly closely the notes “symplectic.pdf” for my March 14 presentation; a lot of that corresponds to the case $k = 1$ in what follows.

The general setting is

(1)(a) $$V = 2n\text{-dimensional vector space over a field } F,$$

(1)(b) $$B = \text{non-degenerate symplectic form on } V.$$

(1)(c) $$Sp(V) = \{g \in GL(V) \mid B(gv, gw) = B(v, w) \quad (v, w \in V)\}.$$

This is the symplectic group of the form B.

The goal is to work out the structure of what are called *maximal parabolic* subgroups of $Sp(V)$, and to look at the corresponding geometry.

Definition 2. An *isotropic subspace* of V is a vector subspace $S \subset V$ with the property that
$$B(v, w) = 0 \qquad (v, w \in S).$$
The corresponding *maximal parabolic subgroup* of $Sp(V)$ is the stabilizer of S in $Sp(V)$:
$$P(S) = \{g \in Sp(V) | g \cdot S = S\}.$$

Definition 3. Suppose $k \geq 0$. The *isotropic Grassmannian* of V is the collection of all k-dimensional isotropic subspaces of V:
$$IG(k, V) = \{S \subset V \mid \dim S = k \text{ and } S \text{ is isotropic}\}.$$
This is a subset of the Grassmann variety $G(k, V)$ of *all* k-dimensional subspaces of V.

The notes on Grassmann varieties worked exclusively with $V = F^n$, and called them $G(k, n)(F)$. Choosing a basis of V identifies $G(k, V)$ with $G(k, 2n)(F)$.

Proposition 4 (Witt). *Suppose that S and S' are k-dimensional isotropic subspaces of the symplectic vector space V. Then there is an element $g \in Sp(V)$ such that $g \cdot S = S'$.*

This is a version of Witt's Extension Theorem, proved in the text for quadratic forms on page 41.

Proof. Choose a basis

(5)(a) $$\{e_1, \ldots, e_k\} \subset S$$

for S. Because U is isotropic,

$$B(e_i, e_j) = 0, \quad (1 \le i, j \le k). \tag{5(b)}$$

Because the u_i are linearly independent, we can find a linear function λ_1 on V with the property that

$$\lambda_1(e_1) = 1, \qquad \lambda_1(e_i) = 0 \quad (2 \le i \le k).$$

By Corollary 2.2 in the text, there is an element $f_1 \in V$ such that

$$\lambda_1(x) = B(x, f_1) \quad (x \in V).$$

That is,

$$B(e_1, f_1) = 1, \qquad B(e_i, f_1) = 0 \quad (2 \le i \le k). \tag{5(c)}$$

Now equations (5)(b) and (5)(c) guarantee that f_1 does not belong to S; so the $k+1$ vectors $\{e_1, \ldots, e_k, f_1\}$ are linearly independent. We may therefore (as long as $k \ge 2$) find a linear function λ_2 on V with the property that

$$\lambda_2(e_1) = \lambda_2(f_1) = 0, \qquad \lambda_2(e_2) = 1, \qquad \lambda_2(e_i) = 0 \quad (3 \le i \le k).$$

By Corollary 2.2 in the text, there is an element $f_2 \in V$ such that

$$\lambda_2(x) = B(x, f_2) \quad (x \in V).$$

That is,

$$B(e_1, f_2) = B(f_1, f_2) = 0, \quad B(e_2, f_2) = 1, \quad B(e_i, f_2) = 0 \quad (3 \le i \le k). \tag{5(d)}$$

Now equations (5)(b), (5)(c), and (5)(d) guarantee that f_2 does not belong to the span of $\{e_1, \ldots, e_k, f_1\}$; so we may continue in the same way. In the end we find k vectors $\{f_1, \ldots, f_k\}$ so that

$$B(e_i, e_j) = B(f_i, f_j) = 0, \qquad B(e_i, f_j) = \delta_{ij} \quad (1 \le i, j \le k). \tag{5(e)}$$

(Here δ_{ij} is the Kronecker delta, equal to 1 if $i = j$ and to zero if 0 if $i \ne j$.) These equations force the $2k$ vectors $\{e_1, \ldots, f_k\}$ to be linearly independent. Write

$$T = \mathrm{span}(f_1, \ldots, f_k). \tag{5(f)}$$

The matrix of the symplectic form B on the $2k$-dimensional subspace $S \oplus T$ is (writing I_k for the $k \times k$ identity matrix, and 0_k for the corresponding zero matrix)

$$\begin{pmatrix} 0_k & -I_k \\ I_k & 0_k. \end{pmatrix}$$

This is invertible, so the restriction of B to $S \oplus T$ is non-degenerate. Define

$$W = (S \oplus T)^\perp = \{w \in V \mid B(w, e_i) = B(w, f_i) = 0 \quad (1 \le i \le k)\}.$$

By Proposition 2.9, we have an orthogonal direct sum decomposition

$$V = (S \oplus T) \oplus W, \tag{5)(g}$$

and the form B_W on W is also non-degenerate. Therefore W is a symplectic vector space of dimension $2n - 2k$.

We can repeat all of these constructions for S', obtaining

$$V = (S' \oplus T') \oplus W'$$

along with special bases $\{e_i'\}$ for S' and $\{f_j'\}$ for T'. Because W and W' are symplectic vector spaces of the same dimension, there is a symplectic isomorphism

$$g_0 \colon W \to W'$$

(text, page 19). We define a linear isomorphism g from V to V by

$$g(e_i) = e_i', \qquad g(f_j) = f_j', \qquad g(w) = g_0(w) \quad (w \in W).$$

From (5)(e) and 5(g), we see that g is a symplectic map, and by construction $g(S) = S'$. $\square$

Corollary 6. *The isotropic Grassmannian $IG(k, V)$ is non-zero if and only if $k \le n$. In that case the action of $Sp(V)$ on $IG(k, V)$ is transitive.*

Proof. The decomposition (5)(g) obtained in the proof of Proposition 4 shows that $2k \le 2n$ whenever there is a k-dimensional isotropic subspace. Conversely, Theorem 2.10 in the text shows that V has a basis $\{u_1, \dots, u_n, v_1, \dots, v_n\}$ such that $B(u_i, u_j) = 0$ for all i and j. Therefore the span of any k-element subset of the $\{u_i\}$ is a k-dimensional isotropic subspace of V. The transitivity is the content of Proposition 4. $\square$

Our next task is to work out the structure of the stabilizer group $P(S)$ for a k-dimensional isotropic subspace S. Just as in the case $k = 1$ treated in the earlier notes, the main step is to write down some interesting elements of $P(S)$. We'll use the basis $\{e_i\}$ of S to identify

$$S \simeq F^k, \tag{7)(a}$$

and similarly the basis $\{f_j\}$ of T to identify

$$T \simeq F^k. \tag{7)(b}$$

Using these identifications and (5)(g), a typical element $v \in V$ may be written as a triple

$$v = (s, t, w) \qquad (s \in F^k, t \in F^k, w \in W) \tag{7)(c}$$

In this picture, (5)(e) and the definition of W show that the symplectic form is

$$B((s_1, t_1, w_1), (s_2, t_2, w_2)) = t_2^{\mathrm{t}} s_1 - t_1^{\mathrm{t}} s_2 + B(w_1, w_2). \tag{7)(d}$$

Here t_2^{t} denotes the transpose of the $k \times 1$ column vector t_2; therefore the matrix product $t_2^{\mathrm{t}} s_1$ is a scalar.

We will describe an element of $P(S)$ by saying first what it does to elements of S (often identified as in (7)(a)), then to elements of T (often identified as in (7)(b)), and finally to elements of $W = (S \oplus T)^{\perp}$. We can then use the formula in (7)(d) to test whether the element so defined respects the symplectic form B.

Here is a first example. Suppose $D \in GL(k, F)$ is an invertible $k \times k$ matrix. Define a linear transformation a_g of V by

$$a_D(s, t, w) = (D \cdot s, (D^{-1})^{\mathrm{t}} \cdot t, w) \tag{8)(a}$$

Using the formula (7)(d), we compute

$$\begin{aligned} B(a_D(s_1, t_1, w_1), a_D(s_2, t_2, w_2)) &= B((Ds_1, (D^{-1})^{\mathrm{t}} t_1, w_1), (Ds_2, (D^{-1})^{\mathrm{t}} t_2, w_2) \\ &= (t_2^{\mathrm{t}} D^{-1})(Ds_1) - (t_1^{\mathrm{t}} D^{-1})(Ds_2) + B(w_1, w_2) \\ &= t_2^{\mathrm{t}} s_1 - t_1^{\mathrm{t}} s_2 + B(w_1, w_2) \\ &= B((s_1, t_1, w_1), (s_2, t_2, w_2)). \end{aligned}$$

Therefore $a_D \in Sp(V)$. Since a_D obviously preserves the subspace S, we have $a_D\ P(S)$.

Next, suppose $R \in Sp(W)$. Define a linear transformation m_r of V by

$$m_R(s, t, w) = (s, t, R \cdot w). \tag{8)(b}$$

(That is, we extend R to V by making it act as the identity on $W^{\perp} = S \oplus T$.) This time the calculation showing that $m_R \in P(S)$ is much simpler, and we omit it. Composition of the first two classes of elements is easy:

$$a_D m_R a_{D'} m_{R'} = a_{gg'} m_{rr'}. \tag{8)(c}$$

For the next class of elements in $P(S)$, we begin with a symmetric $k \times k$ matrix E, and define

$$z_E(s, t, w) = (s + Et, t, w). \tag{8)(d}$$

That z_E preserves S is clear. To check that z_E is symplectic, we compute

$$\begin{aligned} B(z_E(s_1, t_1, w_1), z_E(s_2, t_2, w_2)) &= B((s_1 + Et_1, t_1, w_1), (s_2 + Et_2, t_2, w_2)) \\ &= t_2^{\mathrm{t}}(s_1 + Et_1) - t_1^{\mathrm{t}}(s_2 + Et_2) + B(w_1, w_2) \\ &= t_2^{\mathrm{t}} s_1 - t_1^{\mathrm{t}} s_2 + B(w_1, w_2) + t_2^{\mathrm{t}} E t_1 - t_1^{\mathrm{t}} E t_2. \end{aligned}$$

The last two terms are transposes of each other (since $E^{\mathrm{t}} = E$) and therefore cancel (since they are 1×1). The first three terms are $B((s_1, t_1, w_1), (s_2, t_2, w_2))$, proving that z_E is symplectic. These elements generalize the elements Z_x of the symplectic notes. The collection

$$Z = \{z_E \mid E \text{ symmetric } k \times k\} \tag{8)(e}$$

is a subgroup of $P(S)$, with group law given by addition of symmetric matrices.

The last class of elements in $P(S)$ are analogous to the elements N_w in the symplectic notes, and also to the maps $\rho_{u,y}$ from chapter 6 of the text and from the notes on Ω. In this setting, the datum is a linear map

$$A\colon T \to W. \tag{9)(a}$$

We will write the corresponding symplectic transformations as $n_A \in P(S)$. The first requirement on n_A is

$$n_A(s) = s \quad (s \in S). \tag{9)(b}$$

We would like to define $n_A(t) = t - A(t)$ for $t \in T$. Unfortunately the resulting subspace $n_A(T)$ is no longer isotropic, so this definition cannot be part of a symplectic transformation. We need to add a correction in S. That is, we are going to define a linear map

$$Q\colon T \to S, \tag{9)(c}$$

(depending on A) and then define

$$n_A(t) = t + A(t) + Q(t). \tag{9)(d}$$

Now the linear map Q is just a $k \times k$ matrix (if we use the identifications $S \simeq F^k$ and $T \simeq F^k$ of (7)(a) and (7)(b)). The requirement that the vectors in (9)(d) span an isotropic subspace can be written (using (7)(d)) as

$$t_2^{\mathrm{t}} Q t_1 - t_1^{\mathrm{t}} Q t_2 = B_W(At_1, At_2). \tag{9)(e}$$

That is,

$$t_2^t (Q - Q^{\mathrm{t}}) t_1 = B_W(At_1, At_2) \qquad (t_i \in T).$$

We now *define* Q to be the unique strictly upper triangular $k \times k$ matrix satisfying (9)(e). A little more explicitly, the (i,j)-entry of Q is

$$Q_{ij} = B_W(Af_j, Af_i) \qquad (1 \le i < j \le k). \tag{9)(f}$$

Finally, we need to define n_A on W. Roughly speaking we would like n_A to be the identity on W. The difficulty is that W is not orthogonal (in the symplectic form) to $n_A(T)$. We therefore need to add another correction in S. That is, we are going to define a linear map

$$C\colon W \to S, \tag{9)(g}$$

(depending on A) and then define

$$n_A(w) = w + C(w). \tag{9)(h}$$

The requirement that $n_A(W)$ be orthogonal to $n_A(T)$ can be written (again using (7)(d)) as

$$t^{\mathrm{t}} C(w) = B_W(A(t), w) \qquad (t \in T, w \in W) \tag{9)(i}$$

This requirement defines C completely in terms of A. Here is how. The linear map A is determined by the k vectors $A(e_i) = w_i \in W$. Each of these vectors w_i defines a linear functional λ_i on W, by the rule

$$\lambda_i(w) = B_W(w_i, w).$$

The k linear functionals λ_i are the k coordinates of the map C from W to $S \simeq F^k$. Explicitly,

$$C(w) = \begin{pmatrix} B_W(A(e_1), w) \\ \vdots \\ B_W(A(e_k), w) \end{pmatrix}. \tag{9)(j}$$

(It is reasonable to call the map C the "transpose" of A, but I will not introduce the formal language to justify that.)

Having defined $Q: T \to S$ and $C: W \to S$, we can now define

$$n_A(s, t, w) = (s + C(w) + Q(t), t, w + A(t)) \tag{9)(k}$$

Verification that n_A is a symplectic map (using (7)(d), (9)(e), and (9)(i)) is straightforward; I'll omit it.

Lemma 10. *The collection*

$$N = \{N_A Z_E \mid A \in \mathrm{Hom}(T, W), E \text{ a } k \times k \text{ symmetric matrix}\}$$

of elements of $P(S)$ is a subgroup. The expression of each element of N as $N_A Z_E$ is unique. The group law is

$$N_A Z_E N_B Z_F = N_{A+B} Z_{E+F+(\text{complicated bits})}.$$

Proof. Well, sometime. $\square$

Proposition 11. *Suppose S is a k-dimensional isotropic subspace of the symplectic space V. Arrange notation as in equations (5) above, and define elements of the stabilizer $P(S)$ as in equations (8) and (9). Then Every element of $g \in \mathrm{Stab}_{Sp(V)}([u])$ has a unique representation as a product*

$$g = m_R A_D n_A Z_E \quad (R \in Sp(W), A \in GL(k, F), A \in \mathrm{Hom}(T, W), E \text{ symmetric}).$$

The subgroups N (Lemma 10) and Z are normal in $P(S)$. The quotient group $P(S)/N$ is isomorphic to the product $L = GL(k, F) \times Sp(W)$.

In the case of finite fields, this description allows us to count elements of $P(S)$, at least if we know how to count elements of $Sp(W)$.