General parabolic subgroups of $Sp(2n)$

Let $V$ be a $2n$-dimensional vector space over a field $F$, $B$ be a non-degenerate symplectic form of $V$, $Sp(2n)$ be the set $\{g \in GL(V) \mid B(gv, gw) = B(v, w) \ (v, w \in V)\}$, $S_1$ be an arbitrary isotropic subspace of $V$, and $\dim(S_1) = k_1$.

For $r \in \mathbb{Z}^+$, we arbitrarily pick subspaces $S_2, S_3, \ldots, S_r$ such that

$$S_r \subset S_{r-1} \subset \ldots \subset S_2 \subset S_1$$

Since, $S_1$ is isotropic, it follows that each $S_i$ ($2 \leq i \leq r$) is also isotropic. We see that this is true because since $S_1$ is isotropic, $B(v, w) = 0$ for all $v, w \in S_1$. And, since each $S_i$ ($2 \leq i \leq r$) is a subspace of $S_1$ then $B(v', w') = 0$ for all $v', w' \in S_i$, thereby making $S_i$ isotropic.

Let $\dim(S_i) = k_i$ ($2 \leq i \leq r$). Then, we know that

$$k_r \leq k_{r-1} \leq \ldots \leq k_2 \leq k_1$$

We now define $S_i^\perp$ to be the orthogonal complement of $S_i$ ($1 \leq i \leq r$). Therefore, it is clear that

$$S_r \subset S_{r-1} \subset \ldots \subset S_2 \subset S_1 \subset S_1^\perp \subset S_2^\perp \subset \ldots \subset S_r^\perp$$

Let $W = S_1^\perp/S_1$ to be a set of cosets of the form $v + S_1$ where $v \in S_1^\perp$. We know that $\dim(W) = \dim(S_1^\perp/S_1) = \dim(S_1^\perp) - \dim(S_1)$. Since $\dim(S_1) = k_1$, we also know that $\dim(S_1^\perp) = \dim(V) - k_1 = 2n - k_1$. Let $k = k_1$. Therefore

$$\dim(W) = \dim(S_1^\perp/S_1)$$

$$= \dim(S_1^\perp) - \dim(S_1)$$

$$= 2n - k - k$$

$$= 2(n - k)$$
A stabilizer $P$ for $S_r \subset S_{r-1} \subset \ldots \subset S_2 \subset S_1$ is defined as

$$P(S_r \subset \ldots \subset S_1) = \{ g \in Sp(V) \mid gS_r = S_r, \ldots, gS_1 = S_1 \}$$

To give a homomorphism $f$ from a group $P$ to a direct product group $G_r \times G_{r-1} \times \ldots \times G_1 \times G_0$ is the same as giving $r + 1$ separate homomorphisms $f_i : P \rightarrow G_i$ where $i = 0, 1, \ldots, r$. The relationship is:

$$f(p) = (f_r(p), f_{r-1}(p), \ldots, f_0(p)) \rightarrow G_r \times \ldots \times G_0$$

We now define the following group homomorphism:

$$f : P(S_r \subset \ldots \subset S_1) \rightarrow GL(S_r) \times GL(S_{r-1}/S_r) \times \ldots \times GL(S_1/S_2) \times Sp(W)$$

as a coordinate homomorphism:

If $g \in P$, then

$$f_r(g) = g|_{S_r} \in GL(S_r)$$

$$f_{r-1}(g) = g|_{S_{r-1}/S_r} \in GL(S_{r-1}/S_r)$$

$$\vdots$$

$$f_1(g) = g|_{S_1/S_2} \in GL(S_1/S_2)$$

$$f_0(g) = g|_{S_1/\tilde{S}_1} \in Sp(S_1/\tilde{S}_1)$$

Note that we use the notation $g|_A$ to mean the restriction of $g$ to the subspace $A$.

Let $N_f$ be the kernel of $f$. We know that the kernel of a group homomorphism is a normal subgroup. Thus, $N_f$ is a normal subgroup of $P(S_r \subset \ldots \subset S_1)$.

Let $N$ be a normal subgroup of $S_r \subset \ldots \subset S_1$ (denoted $N(S_r \subset \ldots \subset S_1)$) which is defined as:

$$g \in Sp(V) \text{ such that }$$

$$g(x_r) = x_r \quad (x_r \in S_r),$$

$$g(x_{r-1}) = x_{r-1} + \text{(an element of } S_r) \quad (x_{r-1} \in S_{r-1}),$$

$$\vdots$$

$$g(x_1) = x_1 + \text{(an element of } S_2) \quad (x_1 \in S_1)$$

$$g(w) = w + \text{(an element of } S_1) \quad (w \in S_1)$$

With the help of the class, I outlined some examples in lecture showing that the above definition defines a normal subgroup of $P(S_r \subset \ldots \subset S_1)$. Furthermore, from the definition of $f$ and $N_f$ above, we see that $N_f = N(S_r \subset \ldots \subset S_1)$. 

It then follows that $f$ defines an inclusion:

$$P(S_r \subset \ldots \subset S_1)/N(S_r \subset \ldots \subset S_1) \rightarrow GL(S_r) \times GL(S_{r-1}/S_r) \times \ldots \times GL(S_1/S_2) \times Sp(W)$$

which is also surjective. Note: the proof of surjectivity is left out of these notes; it involves finding a number of elements of $P$ and showing that every element of $GL(S_r) \times GL(S_{r-1}/S_r) \times \ldots \times GL(S_1/S_2) \times Sp(W)$ is an image (under $f$) of elements of $P/N$.

Therefore, $f$ is a bijective map from $P(S_r \subset \ldots \subset S_1)/N(S_r \subset \ldots \subset S_1)$ to $GL(S_r) \times GL(S_{r-1}/S_r) \times \ldots \times GL(S_1/S_2) \times Sp(W)$, which implies that

$$P(S_r \subset \ldots \subset S_1)/N(S_r \subset \ldots \subset S_1) \cong GL(S_r) \times GL(S_{r-1}/S_r) \times \ldots \times GL(S_1/S_2) \times Sp(W)$$

Note the difference between this result and the one demonstrated for maximal parabolic subgroups of $Sp(2n)$ in a previous talk. Specifically, for maximal parabolic subgroups:

$$P(S)/N = GL(k,F) \times S(W)$$

whereas for general parabolic subgroups, we have

$$P(S_r \subset \ldots \subset S_1)/N(S_r \subset \ldots \subset S_1) \cong GL(S_r) \times GL(S_{r-1}/S_r) \times \ldots \times GL(S_1/S_2) \times Sp(W)$$