Introduction The goal of this short presentation is to describe the behavior of several small, classical groups including $\mathsf{PGL}(2,\mathbb{F}_2)$, $\mathsf{PGL}(2,\mathbb{F}_3)$, and $\mathsf{PGL}(2,\mathbb{F}_4)$.

Earlier During a previous seminar, the following result was shown:

$$|\mathsf{GL}(n,q)| = q^{\frac{n(n-1)}{2}} \prod_{i=1}^{n}(q^i - 1) \tag{1}$$

Lemma Suppose q is a prime.

$|\mathsf{GL}(2,q)| = q(q^2-1)(q-1)$

$|\mathsf{PGL}(2,q)| = (q+1)(q)(q-1)$

Recall that $|S_{q+1}| = (q+1)!$. If $q=2$ or $q=3$, then $(q+1)! = (q+1)(q)(q-1)$ because $0! = 1$ and $1! = 1$ respectively. In these two cases, $|\mathsf{PGL}(2,2)| = |S_3|$ and $|\mathsf{PGL}(2,3)| = |S_4|$.

Earlier Recall that in an earlier lecture, it was shown the following groups are isomorphic:

$$\mathsf{PGL}(2,\mathbb{F}_2) \cong \mathsf{GL}(2,\mathbb{F}_2) \cong \mathsf{SL}(2,\mathbb{F}_2) \cong \mathsf{PSL}(2,\mathbb{F}_2) \tag{2}$$

These groups are all isomorphic because they each contain the same six matrices:

$$\left\{ \begin{pmatrix}1&0\\0&1\end{pmatrix} \begin{pmatrix}0&1\\1&0\end{pmatrix} \begin{pmatrix}1&0\\1&1\end{pmatrix} \begin{pmatrix}1&1\\0&1\end{pmatrix} \begin{pmatrix}0&1\\1&1\end{pmatrix} \begin{pmatrix}1&1\\1&0\end{pmatrix} \right\} \tag{3}$$

Let $P'(\mathbb{F}_q^2)$ be the projective line over the field $\mathbb{F}_q$. For the field $\mathbb{F}_q$ the projective line contains the elements:

$$\left\{ \begin{pmatrix}0\\1\end{pmatrix} \begin{pmatrix}1\\0\end{pmatrix} \begin{pmatrix}1\\1\end{pmatrix} \right\} \tag{4}$$

With the operation of left multiplication, the above matrices (3) act on the projective line by permuting its elements. Some matrices switch only two basis elments and leave the third alone, and some act on the projective line by creating a subgroup of order 3. Since $\mathsf{PGL}(2,\mathbb{F}_2)$ permutes these three elements, we have an inclusion (an injection) as follows:

$$\mathsf{PGL}(2,\mathbb{F}_2) \hookrightarrow \mathrm{Perm}\,(P'(\mathbb{F}_2^2)) \cong S_3 \tag{5}$$

There is an isomorphism between the permutation group of the projective line elements and S_3 because both are the group of all permutations over three elements. Therefore,

$$\mathsf{PGL}(2,\mathbb{F}_2) \cong S_3 \tag{6}$$

Lemma For any prime p, there are $p+1$ elements in the projective line with the following form:

$$P'(\mathbb{F}_p^2) = \left\{ \begin{pmatrix}1\\0\end{pmatrix} \begin{pmatrix}1\\1\end{pmatrix} \cdots \begin{pmatrix}1\\(p-1)\end{pmatrix} \begin{pmatrix}0\\1\end{pmatrix} \right\} \tag{7}$$

Description of $\mathsf{PGL}(2, \mathbb{F}_3)$ Because 3 is a prime number we know from the above lemma that $P'(\mathbb{F}_q^2)$ has four elements. Similarly to $\mathsf{PGL}(2, \mathbb{F}_2)$ the action of $\mathsf{PGL}(2, \mathbb{F}_3)$ permutes these four elements creating an inclusion:

$$\mathsf{PGL}(2, \mathbb{F}_3) \hookrightarrow \text{Perm}\,(P'(\mathbb{F}_3^2)) \cong S_4 \tag{8}$$

Since both $\mathsf{PGL}(2, \mathbb{F}_3)$ and S_4 have exactly 24 elements (from first lemma) and an inclusion exists, we know that we must have an isomorphism between the groups

$$\mathsf{PGL}(2, \mathbb{F}_3) \cong S_4 \tag{9}$$

Description of $\mathsf{PGL}(2, \mathbb{F}_4)$ Now we will examine $\mathsf{PGL}(2, \mathbb{F}_3)$ which has $(4+1)(4)(4-1) = 60$ elements. The projective line over $\mathbb{F}_4$ contains the following five elements:

$$\text{Perm}\,(P'(\mathbb{F}_3^2)) = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ a \end{pmatrix} \begin{pmatrix} 0 \\ (1+a) \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \tag{10}$$

These five elements of the projective plane will be referred to as $\overline{0}$, $\overline{1}$, $\overline{a}$, $\overline{1+a}$, and $\overline{\infty}$ respectivefully in the following text. As in a previous lecture on finite fields, a is the elment of $\mathbb{F}_4$ which is the root of $x^2 + x + 1$.

Similar to the above cases, we know that there exists an inclusion based on the fact that the projective linear group permutes the five elements of the projective line in some way:

$$\mathsf{PGL}(2, \mathbb{F}_4) \hookrightarrow \text{Perm}\,(P'(\mathbb{F}_4^2)) \cong S_5 \tag{11}$$

However, in this case the orders of the two groups are different. We have $|S_5| = 120$. As discussed above $|\mathsf{PGL}(2, \mathbb{F}_3)| = 60$. Therefore, there cannot possibly be an isomorphism between groups. For any symmetric group, S_n of order $2k$, there is only one subgroup of order k: The alternating group A_n. Therefore, due to the above inclusion, we must have the following isomorphism:

$$\mathsf{PGL}(2, \mathbb{F}_4) \cong A_5 \tag{12}$$

The Geometric Perspective of $\mathsf{PGL}(2, \mathbb{F}_4)$ Because of the above isomorphism, we expect any elment of $\mathsf{PGL}(2, \mathbb{F}_4)$ act as an even number of transpositions of projective line elements (the alternating group is the subset of the symmetric group consisting of only permutations which can be written as an even number of transpositions). Consider the action of the following three subgroups of $\mathsf{PGL}(2, \mathbb{F}_4)$ on the projective line. For the following subgroups, $x \in \{0, 1, a, a+1\}$ and $y \in \{0, 1, a, a+1\}$. Note that in the second subgroup, $x \neq 0$

1. The Action of $\left\{ \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \right\}$

$\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ The action of matrices in this group fixes $\overline{\infty}$.

$\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \begin{pmatrix} 1 \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ (x+y) \end{pmatrix}$ When an element of the group is considered as a transformation T, we have a map that sends any element y back to itself eventually as follows:

$$y \xrightarrow{T} y + x \xrightarrow{T} y + 2x = y \tag{13}$$

Recall that in $\mathbb{F}_4$, $2 = 0$. Therefore the elements of N fix $\overline{\infty}$, and have two transpositions $(y, y + x)$ and the other two remaining elements, which are mapped back and forth by the addition of x. Therefore, elements of this form act as two (or none) transpositions. Elements of this subgroup have an even number of transpositions.

2. The Action of $\left\{\begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}\right\}$ This group contains four matrices (for the four possible values of x) and is isomorphic to the kline four group. This group will be reffered to as N.

$\begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ The action fixes $\overline{0}$

$\begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ x \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ The action fixes $\overline{\infty}$.

$\begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}\begin{pmatrix} 1 \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ xy \end{pmatrix}$ When an element of N is considered as a transformation T, we have a map that sends any element y back to itself eventually as follows:

$$y \xrightarrow{T} xy \xrightarrow{T} x^2y \xrightarrow{T} x^3y = y \tag{14}$$

Recall that in $|\mathbb{F}_4^\times| = 3$, so the any element to the power of three is 1. Therefore this elements of this subgroup fix $\overline{\infty}$ and $\overline{0}$, and create a 3-cycle with the remaining elements. A 3-cycle can be written as two transpositions. Elements of this subgroup have an even number of transpositions.

3. The Action of $\left\{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\}$

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ The action transposes $\overline{0}$ and $\overline{\infty}$

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ The action fixes $\overline{1}$

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ a \end{pmatrix} = \begin{pmatrix} a \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ a+1 \end{pmatrix}$ The action transposes $\overline{a}$ and $\overline{a+1}$.

Therefore, the action can be written as two transpositions.

Conclusion Using knowlege from linear algebra, every invertible matrix can be written the product of elementary matrices which each have a form belonging to one of the three subgroups above (the first being addition of rows, the second being multiplication of a row by a scalar, and the third being the switching of two rows). Every matrix in $\mathsf{PGL}(2, \mathbb{F}_4)$ can be generated by the elements of these three subgroups. Every matrix of $\mathsf{PGL}(2, \mathbb{F}_4)$ can be decomposed into an even number of transpositions on the five elements of Perm $(P'(\mathbb{F}_3^2))$. Therefore we have an inclusion of

$$\mathsf{PGL}(2, \mathbb{F}_4) \hookrightarrow A_5. \tag{15}$$

Since both groups have order 60, there must be an isomorphism.

$$\mathsf{PGL}(2, \mathbb{F}_4) \cong A_5. \tag{16}$$