Representations of $\mathfrak{sl}(2)$

March 28, 2015

1 Representations of $\mathfrak{sl}(2)$

{sec:sl2rep}

The point of these notes is to summarize and complete what was done in class March 19 about the representation theory of the three-dimensional Lie algebra

$$\mathfrak{g} = \mathfrak{sl}(2,k) = 2 \times 2 \text{ matrices over } k \text{ of trace zero.}$$

This Lie algebra has a basis consisting of the three matrices

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

(That this is a basis follows immediately from the fact that the four "matrix units" e_{ij}, with $1 \le i, j \le 2$, are a basis for the vector space of *all* matrices. You want a basis for the kernel of the linear map tr to k.)

Here is the main theorem about representations.

{thm:sl2rep}

Theorem 1.1. *[text, Theorem 4.59] For each integer $m \ge 0$ there is a representation $(\rho_m, V(m))$ of $\mathfrak{sl}(2,k)$ on a vector space of dimension $m+1$ over k, defined as follows. The space $V(m)$ has a basis*

$$\{v_{m-2j} \mid 0 \le j \le m\} = \{v_m, v_{m-2}, \ldots, v_{-m}\}.$$

The action of the basis elements of the Lie algebra is given by

$$\begin{aligned} \rho_m(H)v_{m-2j} &= (m-2j)v_{m-2j} \\ \rho_m(X)v_{m-2j} &= jv_{m-2j+2} \qquad\qquad (0 \le j \le m) \\ \rho_m(Y)v_{m-2j} &= (m-j)v_{m-2j-2}. \end{aligned}$$

Equivalently,

$$\begin{aligned}
\rho_m(H)v_p &= pv_p \\
\rho_m(X)v_p &= \frac{m-p}{2}v_{p+2} \qquad\qquad (p = m, m-2, \ldots, -m) \\
\rho_m(Y)v_p &= \frac{m+p}{2}v_{p-2}.
\end{aligned}$$

As long as the characteristic of k is either zero or strictly larger than m, the representation ρ_m is irreducible.

Suppose now that $\operatorname{char} k = 0$.

1. *Every irreducible representation of $\mathfrak{sl}(2,k)$ is isomorphic to some $V(m)$.*

2. *Every finite-dimensional representation of $\mathfrak{sl}(2,k)$ is isomorphic to a direct sum of copies of $V(m)$.*

3. *The element H acts diagonalizably with integer eigenvalues in any finite-dimensional representation of $\mathfrak{sl}(2,k)$.*

Before beginning the proof of the theorem, we need some additional notation. The Lie bracket is commutator of matrices, so we compute {se:sl2notation}

$$[H,X] = 2X, \qquad [H,Y] = -2Y, \qquad [X,Y] = H. \tag{1.2a}$$ {e:sl2rel}

These bracket relations describe the matrices $\operatorname{ad}(X)$, $\operatorname{ad}(H)$, and $\operatorname{ad}(Y)$ in the basis $\{X, H, Y\}$:

$$\operatorname{ad}(X) = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \ \operatorname{ad}(H) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \ \operatorname{ad}(Y) = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}. \tag{1.2b}$$

From these matrices one immediately calculates the Killing form

$$B(H,H) = 4, \qquad B(X,Y) = 4, \qquad B(X,H) = B(Y,H) = 0. \tag{1.2c}$$

The Killing form is nondegenerate as long as $\operatorname{char} k \neq 2$. In that case the dual basis to $\{X, H, Y\}$ is $\{\frac{1}{2}Y, \frac{1}{4}H, \frac{1}{2}X\}$, so the Casimir operator is

$$C_B = \frac{1}{2}(XY + YX) + \frac{1}{4}H^2 = \frac{1}{4}(H^2 + 2XY + 2YX) \in U(\mathfrak{g}).$$

I proved in class (see also the text, Proposition 6.15) that C_B belongs to

$$\mathfrak{Z}(\mathfrak{g}) =_{\text{def}} \text{center of } U(\mathfrak{g}). \tag{1.2d}$$

Using the defining relation

$$XY = YX + [X, Y] = YX + H$$

of the enveloping algebra, we calculate (still with char $k \neq 2$)

$$C_B = \frac{1}{4}(H^2 + 2H + 4YX) = \frac{1}{4}(H^2 - 2H + 4XY) \in \mathfrak{Z}(\mathfrak{g}). \tag{1.2e}$$

It is convenient (and fairly standard) to multiply this element by four, and to define

$$\Omega = H^2 + 2H + 4YX = H^2 - 2H + 4XY \in \mathfrak{Z}(\mathfrak{g}). \tag{1.2f}$$ {e:omega}

The element Ω is defined for any field k (even in characteristic two, where it reduces to H^2).

{se:sl2proof}

Proof of Theorem 1.1. To prove that ρ_m is a representation, one has to do three calculations like this:

$$\begin{aligned}
[\rho_m(H), \rho_m(X)] &= (\rho_m(H)\rho_m(X) - \rho_m(X)\rho_m(H))\, v_{m-2j} \\
&= \rho_m(H)(j \cdot v_{m-2j+2}) - \rho_m(X)((m - 2j) \cdot v_{m-2j}) \\
&= j(m - 2j + 2)v_{m-2j+2} - j(m - 2j)v_{m-2j+2} \\
&= 2jv_{m-2j+2} \\
&= \rho_m(2X)v_{m-2j}.
\end{aligned}$$

Since this equality holds for every basis vector v_{m-2j}, the conclusion is that

$$[\rho_m(H), \rho_m(X)] = \rho_m(2X) = \rho_m([H, X]). \tag{1.3a}$$

Together with parallel calculations for $[H, Y]$ and $[X, Y]$, this shows that ρ_m respects the Lie bracket, and so is a representation.

Suppose now that char(k) is zero or greater than m; we want to show that ρ_m is irreducible. Suppose $W \subset V(m)$ is a nonzero invariant subspace. Because $\rho_m(X)$ is obviously a nilpotent linear map, the restriction of $\rho_m(X)$ to W is also nilpotent, and so must contain a nonzero vector w in the kernel of $\rho_m(X)$. But the formula in the theorem for $\rho_m(X)$ shows that

$$\ker(\rho_m(X)) = \operatorname{span}\{v_{m-2j} \mid j = 0 \text{ in } k\} = kv_m,$$

the last equality because of the assumption that char(k) is zero or greater than m. The conclusion is that $v_m \in W$. Applying $\rho_m(Y)$ to v_m repeatedly (and using again the hypothesis on char(k)) we find that W contains every basis vector v_{m-2j}; so $W = V(m)$, as we wished to show.

We record also the fact

$$\rho_m(\Omega) = (m^2 + 2m)I. \tag{1.3b}$$ {e:omegaeigenvalue}

When ρ_m is irreducible and k is algebraically closed, Schur's lemma tells us that $\rho_m(\Omega)$ must be a scalar. We can compute that scalar most easily by applying Ω to v_m and using the first formula in (1.2f): $\rho_m(X)v_m = 0$, so the calculation is very easy. It's clear that knowing (1.3b) for the algebraic closure of k implies it for k. To get the identity in fields of small characteristic, one can either calculate the action of $\rho_m(\Omega)$ on every basis vector v_{m-2j}, or use the fact that the identity is true "over $\mathbb{Z}$."

Because $(\rho_m(H^2 - 2H)(v_p) = (p^2 - 2p)v_p$, we calculate

$$\begin{aligned}\rho_m(4XY)v_{m-2j} &= \rho_m(\Omega - H^2 + 2H)v_{m-2j}\\ &= (m^2 + 2m - (m-2j)^2 + 2(m-2j))v_{m-2j}\\ &= (4jm + 4m - 4j^2 - 4j)v_{m-2j} = 4(j+1)(m-j)v_{m-2j}.\end{aligned}$$

As long as $\operatorname{char} k \neq 2$, we can divide by four to get

$$\begin{aligned}\rho_m(XY)v_{m-2j} &= (j+1)(m-j)v_{m-2j},\\ \rho_m(XY)v_p &= \frac{(m-p+2)}{2}\frac{m+p}{2}v_p.\end{aligned} \tag{1.3c}$$ {e:XY}

This equation is still true in characteristic two, either by direct calculation from the formulas in the theorem or by another argument about identities over $\mathbb{Z}$.

Now suppose that (τ, W) is a finite-dimensional representation of $\mathfrak{sl}(2,k)$. For any $\lambda \in k$, consider the λ eigenspace of $\tau(H)$:

$$W_\lambda =_{\text{def}} \{w \in W \mid \tau(H)w = \lambda w\}. \tag{1.3d}$$

It is an immediate consequence of (1.2a) that

$$\tau(X)W_\lambda \subset W_{\lambda+2}, \qquad \tau(Y)W_\lambda \subset W_{\lambda-2}. \tag{1.3e}$$ {e:sl2wt}

If we define

$$W_{\lambda+2\mathbb{Z}} = \sum_{j\in\mathbb{Z}} W_{\lambda+2j}, \tag{1.3f}$$

then the conclusion is that $W_{\lambda+2\mathbb{Z}}$ is an invariant subspace of W.

In the same way, for any $\kappa \in k$, consider the κ eigenspace of $\tau(\Omega)$:

$$W^\kappa =_{\text{def}} \{w \in W \mid \tau(\Omega)w = \kappa w\}. \tag{1.3g}$$

Because $\Omega \in \mathfrak{Z}(\mathfrak{g})$, any such eigenspace is an invariant subspace of W.

Assume for a moment that k is algebraically closed, and that $W \neq 0$. Then $\tau(H)$ must have an eigenvalue; that is,

$$W_\lambda \neq 0 \qquad (\text{some } \lambda \in k).$$

Such λ are called *weights* of W. Because k has characteristic zero, the various $\lambda + 2j$ (for $j \in \mathbb{Z}$) are *distinct* elements of k. Since W is finite-dimensional, these cannot all be eigenvalues of $\tau(H)$. The conclusion is

$$W_\mu \neq 0, \quad W_{\mu+2} = 0 \qquad (\text{some } \mu \in k). \tag{1.3h}$$ {e:hwt}

Such a μ is called a *highest weight* of W. Combining (1.3e) with (1.3h), we find

$$\tau(H)w = \mu w, \quad \tau(X)w = 0 \qquad (w \in W_\mu). \tag{1.3i}$$

In light of the formula (1.2f), we conclude that

$$\tau(\Omega)w = (\mu^2 + 2\mu)w = [(\mu+1)^2 - 1]w \qquad (w \in W_\mu). \tag{1.3j}$$ {e:cashwt}

We have now found a nonzero invariant subspace

$$S = W^{\mu^2+2\mu}_{\mu+2\mathbb{Z}} \subset W, \qquad S_\mu = W_\mu. \tag{1.3k}$$ {e:subspace}

Lemma 1.4. *Suppose* $\operatorname{char} k = 0$, $\mu \in k$, *and* (σ, S) *is a nonzero finite-dimensional representation of* $\mathfrak{sl}(2,k)$ *satisfying*

1. $S_\mu \neq 0$, $S_{\mu+2} = 0$; *and*
2. $\sigma(\Omega) = (\mu^2 + 2\mu)I$.

Then $\mu = m$ *is a nonnegative integer, and* σ *contains* $(\rho_m, V(m)$ *as a subrepresentation.*

{se:sl2lemmaproof}

Proof. We can find in S a *lowest* weight for H in the same way we found a highest weight: a weight $\mu - 2m$ so that

$$S_{\mu-2m} \neq 0, \quad S_{\mu-2m-2} = 0 \qquad (\text{some } m \in \mathbb{N}). \tag{1.5a}$$ {e:lwt}

Calculating the action of Ω exactly as in (1.3j) gives

$$\sigma(\Omega)u = (\mu-2m)^2 - 2(\mu-2m))u = [(\mu-2m-1)^2 - 1]u \qquad (u \in S_{\mu-2m}). \tag{1.5b}$$ {e:caslwt}

By hypothesis, Ω acts by the scalar $(\mu+1)^2-1$. The conclusion is

$$(\mu+1)^2-1=(\mu-2m-1)^2-1,$$

or

$$2\mu=-4m\mu-2\mu+4m^2$$

or

$$4\mu(m+1)=4m(m+1).$$

Since m is a nonnegative integer and $\operatorname{char} k=0$, we can conclude that $\mu=m$.

Now choose any nonzero vector $v_m\in S_\mu=S_m$, and define

$$v_{m-2j}=\frac{1}{m(m-1)\cdots(m-(j-1))}\sigma(Y)^j v_m \qquad (0\le j\le m). \tag{1.5c}$$

The numerator here is a product of j factors, each of which is nonzero since $\operatorname{char} k=0$. Because of (1.3e) and this definition, we calculate immediately

$$\sigma(H)v_{m-2j}=(m-2j)v_{m-2j}, \qquad \sigma(Y)v_{m-2j}=(m-j)v_{m-2j-2}. \tag{1.5d}$$

(For the last equality in case $j=m$, we use also the fact that $S_{-m-2}=S_{\mu-2m-2}=0$ by the choice of m.) These are two of the three defining relations of ρ_m from the theorem.

For the last, part of the hypothesis on S is that

$$\sigma(\Omega)w=(\mu^2+2\mu)w=(m^2+2m)w \qquad (w\in S).$$

Exactly the same proof as we gave for (1.3c) therefore shows that

$$\sigma(XY)v_{m-2j'}=(j'+1)(m-j')v_{m-2j'} \qquad (0\le j'\le m). \tag{1.5e}$$

Inserting the formula for the action of $\tau(Y)$ gives

$$\sigma(X)(m-j')v_{m-2j'-2}=(j'+1)(m-j')v_{m-2j'} \qquad (0\le j'\le m-1).$$

The factor $m-j'$ is nonzero in this range, so we get

$$\sigma(X)v_{m-2j'-2}=(j'+1)v_{m-2j'},$$

or (writing $j=j'+1$)

$$\sigma(X)v_{m-2j}=jv_{m-2j+2} \qquad (1\le j\le m). \tag{1.5f}$$

The same formula holds also for $j=0$ because of the highest weight hypothesis $S_{\mu+2}=S_{m+2}=0$. $\square$

The lemma completes the proof of (1) in the theorem in case k is algebraically closed. Part (2) is Weyl's theorem of complete reducibility, and (3) is immediate from (2).

Suppose finally that k has characteristic zero but is not algebraically closed, and (τ, W) is any finite-dimensional representation of $\mathfrak{sl}(2, k)$. If $\overline{k}$ is an algebraic closure of k, then

$$W_{\overline{k}} =_{\text{def}} \overline{k} \otimes_k W$$

is a finite-dimensional representation $\overline{\tau}$ of $\mathfrak{sl}(2, \overline{k})$; the operators $\overline{\tau}(X)$, $\overline{\tau}(H)$, and $\overline{\tau}(Y)$ may be represented by exactly the same k-matrices as for τ, just interpreted as $\overline{k}$-matrices. By the algebraically closed case, $\overline{\tau}(H)$ is diagonalizable with integer eigenvalues. Because these eigenvalues belong to $\mathbb{Z} \subset k$, $\tau(H)$ is also diagonalizable with integer eigenvalues.

We used the algebraically closed assumption above only to find an eigenvalue of $\tau(H)$; and now we have that for any field of characteristic zero. The construction of S, and the inclusion of $V(m)$ in S, proceeds exactly as above. $\square$