Representations of $\mathfrak{sl}(2)$

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1 Representations of $\mathfrak{sl}(2)$

The point of these notes is to summarize and complete what was done in class March 19 about the representation theory of the three-dimensional Lie algebra

$$\mathfrak{g} = \mathfrak{sl}(2, k) = 2 \times 2 \text{ matrices over } k \text{ of trace zero.}$$

This Lie algebra has a basis consisting of the three matrices

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$  

(That this is a basis follows immediately from the fact that the four “matrix units” $e_{ij}$, with $1 \leq i, j \leq 2$, are a basis for the vector space of all matrices. You want a basis for the kernel of the linear map $tr$ to $k$.)

Here is the main theorem about representations.

**Theorem 1.1.** [text, Theorem 4.59] For each integer $m \geq 0$ there is a representation $(\rho_m, V(m))$ of $\mathfrak{sl}(2, k)$ on a vector space of dimension $m+1$ over $k$, defined as follows. The space $V(m)$ has a basis

$$\{v_{m-2j} \mid 0 \leq j \leq m\} = \{v_m, v_{m-2}, \ldots, v_{-m}\}.$$  

The action of the basis elements of the Lie algebra is given by

$$\rho_m(H)v_{m-2j} = (m - 2j)v_{m-2j}$$
$$\rho_m(X)v_{m-2j} = jv_{m-2j+2} \quad (0 \leq j \leq m)$$
$$\rho_m(Y)v_{m-2j} = (m - j)v_{m-2j-2}.$$
Equivalently,
\[
\begin{align*}
\rho_m(H)v_p &= pv_p \\
\rho_m(X)v_p &= \frac{m-p}{2}v_{p+2} \quad (p = m, m - 2, \ldots, -m) \\
\rho_m(Y)v_p &= \frac{m+p}{2}v_{p-2}.
\end{align*}
\]

As long as the characteristic of \( k \) is either zero or strictly larger than \( m \), the representation \( \rho_m \) is irreducible.

Suppose now that \( \text{char } k = 0 \).

1. Every irreducible representation of \( \mathfrak{sl}(2, k) \) is isomorphic to some \( V(m) \).

2. Every finite-dimensional representation of \( \mathfrak{sl}(2, k) \) is isomorphic to a direct sum of copies of \( V(m) \).

3. The element \( H \) acts diagonalizably with integer eigenvalues in any finite-dimensional representation of \( \mathfrak{sl}(2, k) \).

Before beginning the proof of the theorem, we need some additional notation. The Lie bracket is commutator of matrices, so we compute
\[
[H, X] = 2X, \quad [H, Y] = -2Y, \quad [X, Y] = H. \quad (1.2a)
\]

These bracket relations describe the matrices \( \text{ad}(X) \), \( \text{ad}(H) \), and \( \text{ad}(Y) \) in the basis \( \{X, H, Y\} \):
\[
\begin{align*}
\text{ad}(X) &= \begin{pmatrix} 0 & -2 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0 \end{pmatrix}, & \text{ad}(H) &= \begin{pmatrix} 2 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & -2 \end{pmatrix}, & \text{ad}(Y) &= \begin{pmatrix} 0 & 0 & 0 \\
-1 & 0 & 0 \\
0 & 2 & 0 \end{pmatrix}. \quad (1.2b)
\end{align*}
\]

From these matrices one immediately calculates the Killing form
\[
B(H, H) = 4, \quad B(X, Y) = 4, \quad B(X, H) = B(Y, H) = 0. \quad (1.2c)
\]

The Killing form is nondegenerate as long as \( \text{char } k \neq 2 \). In that case the dual basis to \( \{X, H, Y\} \) is \( \{\frac{1}{2}Y, \frac{1}{4}H, \frac{1}{2}X\} \), so the Casimir operator is
\[
C_B = \frac{1}{2}(XY + YX) + \frac{1}{4}H^2 = \frac{1}{4}(H^2 + 2XY + 2YX) \in U(\mathfrak{g}).
\]

I proved in class (see also the text, Proposition 6.15) that \( C_B \) belongs to
\[
\mathfrak{z}(\mathfrak{g}) =_{\text{def}} \text{center of } U(\mathfrak{g}). \quad (1.2d)
\]
Using the defining relation

\[ XY = YX + [X,Y] = YX + H \]

of the enveloping algebra, we calculate (still with char \( k \neq 2 \))

\[ C_B = \frac{1}{4} (H^2 + 2H + 4YX) = \frac{1}{4} (H^2 - 2H + 4XY) \in \mathfrak{z}(\mathfrak{g}). \quad (1.2e) \]

It is convenient (and fairly standard) to multiply this element by four, and to define

\[ \Omega = H^2 + 2H + 4YX = H^2 - 2H + 4XY \in \mathfrak{z}(\mathfrak{g}). \quad (1.2f) \]

The element \( \Omega \) is defined for any field \( k \) (even in characteristic two, where it reduces to \( H^2 \)).

**Proof of Theorem 1.1.** To prove that \( \rho_m \) is a representation, one has to do three calculations like this:

\[
[\rho_m(H), \rho_m(X)] = (\rho_m(H)\rho_m(X) - \rho_m(X)\rho_m(H)) v_{m-2j} \\
= \rho_m(H)(j \cdot v_{m-2j+2}) - \rho_m(X)((m - 2j) \cdot v_{m-2j}) \\
= j(m - 2j + 2)v_{m-2j+2} - j(m - 2j)v_{m-2j+2} \\
= 2j v_{m-2j+2} \\
= \rho_m(2X)v_{m-2j}.
\]

Since this equality holds for every basis vector \( v_{m-2j} \), the conclusion is that

\[ [\rho_m(H), \rho_m(X)] = \rho_m(2X) = \rho_m([H,X]). \quad (1.3a) \]

Together with parallel calculations for \([H,Y]\) and \([X,Y]\), this shows that \( \rho_m \) respects the Lie bracket, and so is a representation.

Suppose now that char\((k)\) is zero or greater than \( m \); we want to show that \( \rho_m \) is irreducible. Suppose \( W \subset V(m) \) is a nonzero invariant subspace. Because \( \rho_m(X) \) is obviously a nilpotent linear map, the restriction of \( \rho_m(X) \) to \( W \) is also nilpotent, and so must contain a nonzero vector \( w \) in the kernel of \( \rho_m(X) \). But the formula in the theorem for \( \rho_m(X) \) shows that

\[ \ker(\rho_m(X)) = \text{span}\{v_{m-2j} \mid j = 0 \text{ in } k\} = kv_m, \]

the last equality because of the assumption that char\((k)\) is zero or greater than \( m \). The conclusion is that \( v_m \in W \). Applying \( \rho_m(Y) \) to \( v_m \) repeatedly
(and using again the hypothesis on char(\(k\))) we find that \(W\) contains every basis vector \(v_{m-2j}\); so \(W = V(m)\), as we wished to show.

We record also the fact

\[ \rho_m(\Omega) = (m^2 + 2m)I. \]  \tag{1.3b} \{e:omegaeigenvalue\}

When \(\rho_m\) is irreducible and \(k\) is algebraically closed, Schur’s lemma tells us that \(\rho_m(\Omega)\) must be a scalar. We can compute that scalar most easily by applying \(\Omega\) to \(v_m\) and using the first formula in (1.2f):

\[ \rho_m(X)v_m = 0, \]

so the calculation is very easy. It’s clear that knowing (1.3b) for the algebraic closure of \(k\) implies it for \(k\). To get the identity in fields of small characteristic, one can either calculate the action of \(\rho_m(\Omega)\) on every basis vector \(v_{m-2j}\), or use the fact that the identity is true “over \(\mathbb{Z}\).”

Because \((\rho_m(H^2 - 2H)(v_p) = (p^2 - 2p)v_p\), we calculate

\[
\rho_m(4XY)v_{m-2j} = \rho_m(\Omega - H^2 + 2H)v_{m-2j} \\
= (m^2 + 2m - (m - 2j)^2 + 2(m - 2j))v_{m-2j} \\
= (4jm + 4m - 4j^2 - 4j)v_{m-2j} = 4(j + 1)(m - j)v_{m-2j}.
\]

As long as \(\text{char } k \neq 2\), we can divide by four to get

\[
\rho_m(XY)v_{m-2j} = (j + 1)(m - j)v_{m-2j}, \\
\rho_m(XY)v_p = \frac{(m - p + 2) m + p}{2} v_p. \tag{1.3c} \{e:XY\}
\]

This equation is still true in characteristic two, either by direct calculation from the formulas in the theorem or by another argument about identities over \(\mathbb{Z}\).

Now suppose that \((\tau, W)\) is a finite-dimensional representation of \(\mathfrak{sl}(2, k)\). For any \(\lambda \in k\), consider the \(\lambda\) eigenspace of \(\tau(H)\):

\[ W_\lambda = \{ w \in W \mid \tau(H)w = \lambda w \}. \tag{1.3d} \]

It is an immediate consequence of (1.2a) that

\[ \tau(X)W_\lambda \subset W_{\lambda+2}, \quad \tau(Y)W_\lambda \subset W_{\lambda-2}. \tag{1.3e} \{e:sl2wt\} \]

If we define

\[ W_{\lambda+2\mathbb{Z}} = \sum_{j \in \mathbb{Z}} W_{\lambda+2j}, \tag{1.3f} \]

then the conclusion is that \(W_{\lambda+2\mathbb{Z}}\) is an invariant subspace of \(W\).
In the same way, for any \( \kappa \in k \), consider the \( \kappa \) eigenspace of \( \tau(\Omega) \):

\[
W^\kappa \overset{\text{def}}{=} \{ w \in W \mid \tau(\Omega)w = \kappa w \}. \tag{1.3g}
\]

Because \( \Omega \in \mathfrak{z}(\mathfrak{g}) \), any such eigenspace is an invariant subspace of \( W \).

Assume for a moment that \( k \) is algebraically closed, and that \( W \neq 0 \). Then \( \tau(H) \) must have an eigenvalue; that is,

\[
W_\lambda \neq 0 \quad (\text{some } \lambda \in k).
\]

Such \( \lambda \) are called weights of \( W \). Because \( k \) has characteristic zero, the various \( \lambda + 2j \) (for \( j \in \mathbb{Z} \)) are distinct elements of \( k \). Since \( W \) is finite-dimensional, these cannot all be eigenvalues of \( \tau(H) \). The conclusion is

\[
W_\mu \neq 0, \quad W_{\mu+2} = 0 \quad (\text{some } \mu \in k). \tag{1.3h}
\]

Such a \( \mu \) is called a highest weight of \( W \). Combining (1.3e) with (1.3h), we find

\[
\tau(H)w = \mu w, \quad \tau(X)w = 0 \quad (w \in W_\mu). \tag{1.3i}
\]

In light of the formula (1.2f), we conclude that

\[
\tau(\Omega)w = (\mu^2 + 2\mu)w = [(\mu + 1)^2 - 1]w \quad (w \in W_\mu). \tag{1.3j}
\]

We have now found a nonzero invariant subspace

\[
S = W^{\mu^2+2\mu}_{\mu+2} \subset W, \quad S_\mu = W_\mu. \tag{1.3k}
\]

**Lemma 1.4.** Suppose \( \text{char } k = 0, \mu \in k, \) and \( (\sigma,S) \) is a nonzero finite-dimensional representation of \( \mathfrak{sl}(2,k) \) satisfying

1. \( S_\mu \neq 0, \quad S_{\mu+2} = 0; \) and
2. \( \sigma(\Omega) = (\mu^2 + 2\mu)I. \)

Then \( \mu = m \) is a nonnegative integer, and \( \sigma \) contains \( (\rho_m,V(m)) \) as a sub-representation.

**Proof.** We can find in \( S \) a lowest weight for \( H \) in the same way we found a highest weight: a weight \( \mu - 2m \) so that

\[
S_{\mu-2m} \neq 0, \quad S_{\mu-2m-2} = 0 \quad (\text{some } m \in \mathbb{N}). \tag{1.5a}
\]

Calculating the action of \( \Omega \) exactly as in (1.3j) gives

\[
\sigma(\Omega)u = (\mu - 2m)^2 - 2(\mu - 2m))u = [(\mu - 2m - 1)^2 - 1]u \quad (u \in S_{\mu-2m}). \tag{1.5b}
\]
By hypothesis, $\Omega$ acts by the scalar $(\mu + 1)^2 - 1$. The conclusion is

$$(\mu + 1)^2 - 1 = (\mu - 2m - 1)^2 - 1,$$

or

$$2\mu = -4m\mu - 2\mu + 4m^2$$

or

$$4\mu(m + 1) = 4m(m + 1).$$

Since $m$ is a nonnegative integer and $\text{char} \ k = 0$, we can conclude that $\mu = m$.

Now choose any nonzero vector $v_m \in S_\mu = S_m$, and define

$$v_{m-2j} = \frac{1}{m(m-1)\cdots(m-(j-1))} \sigma(Y)^j v_m \quad (0 \leq j \leq m). \quad (1.5c)$$

The numerator here is a product of $j$ factors, each of which is nonzero since $\text{char} \ k = 0$. Because of (1.3e) and this definition, we calculate immediately

$$\sigma(H)v_{m-2j} = (m-2j)v_{m-2j}, \quad \sigma(Y)v_{m-2j} = (m-j)v_{m-2j-2}. \quad (1.5d)$$

(For the last equality in case $j = m$, we use also the fact that $S_{m-2} = S_{\mu-2m-2} = 0$ by the choice of $m$.) These are two of the three defining relations of $\rho_m$ from the theorem.

For the last, part of the hypothesis on $S$ is that

$$\sigma(\Omega)w = (\mu^2 + 2\mu)w = (m^2 + 2m)w \quad (w \in S).$$

Exactly the same proof as we gave for (1.3c) therefore shows that

$$\sigma(XY)v_{m-2j'} = (j'+1)(m-j')v_{m-2j'} \quad (0 \leq j' \leq m). \quad (1.5e)$$

Inserting the formula for the action of $\tau(Y)$ gives

$$\sigma(X)(m-j')v_{m-2j'-2} = (j'+1)(m-j')v_{m-2j'} \quad (0 \leq j' \leq m-1).$$

The factor $m-j'$ is nonzero in this range, so we get

$$\sigma(X)v_{m-2j'-2} = (j'+1)v_{m-2j'},$$

or (writing $j = j' + 1$)

$$\sigma(X)v_{m-2j} = jv_{m-2j+2} \quad (1 \leq j \leq m). \quad (1.5f)$$

The same formula holds also for $j = 0$ because of the highest weight hypothesis $S_{\mu+2} = S_{m+2} = 0$. \qed
The lemma completes the proof of (1) in the theorem in case $k$ is algebraically closed. Part (2) is Weyl’s theorem of complete reducibility, and (3) is immediate from (2).

Suppose finally that $k$ has characteristic zero but is not algebraically closed, and $(\tau, W)$ is any finite-dimensional representation of $\mathfrak{sl}(2,k)$. If $\overline{k}$ is an algebraic closure of $k$, then

$$W_{\overline{k}} = \text{def} \overline{k} \otimes_k W$$

is a finite-dimensional representation $\tau$ of $\mathfrak{sl}(2,\overline{k})$; the operators $\tau(X), \tau(H)$, and $\tau(Y)$ may be represented by exactly the same $k$-matrices as for $\tau$, just interpreted as $\overline{k}$-matrices. By the algebraically closed case, $\tau(H)$ is diagonalizable with integer eigenvalues. Because these eigenvalues belong to $\mathbb{Z} \subset k$, $\tau(H)$ is also diagonalizable with integer eigenvalues.

We used the algebraically closed assumption above only to find an eigenvalue of $\tau(H)$; and now we have that for any field of characteristic zero. The construction of $S$, and the inclusion of $V(m)$ in $S$, proceeds exactly as above. \qed