The size of infinite-dimensional representations II

David Vogan

Department of Mathematics
Massachusetts Institute of Technology

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Outline

Introduction

Geometrizing representations

Equivariant K-theory

K-theory and representations

Complex groups: ∞-diml reps and algebraic geometry

Lusztig's conjecture and generalizations

Slides at `http://www-math.mit.edu/~dav/paper.html`

Where we (should have) ended yesterday

$G = GL(n,\mathbb{R})$, $\theta(g) = {}^t g^{-1}$ Cartan involution.

$K = GL(n,\mathbb{R})^\theta = O(n)$ (compact, easy).

$\Delta_G = 2\Omega_K - \Omega_G \in U_2(\mathfrak{g})$ difference of Casimir ops.

$(\pi, \mathcal{H}_\pi) \in \widehat{G}$; eigval aymptotics of $\pi^\infty(\Delta_G) \rightsquigarrow \mathrm{Dim}(\pi)$.

Start today by modifying point of view:

$$\mathcal{H}_\pi = \sum_{\mu \in \widehat{O(n)}} \mathcal{H}_\pi(\mu) \simeq \sum m_\pi(\mu)\mu \qquad (m_\pi(\mu) \in \mathcal{N}).$$

Since $\pi^\infty(\Omega_G) = c(\pi) \in \mathbb{R}$,

eigval asymp of $\Delta_G =$ asymp of restr to K.

If $\mathcal{H}_\pi(N) =_{\mathrm{def}} \sum_{\mu(\Omega_K) \le N^2} \mathcal{H}_\pi(\mu)$, then

$$\dim \mathcal{H}_\pi(N) \sim a(\pi) N^{\mathrm{Dim}(\pi)}.$$

Understanding size means understanding $\pi|_K$.

Stating the question and changing notation

Two goals today:

1. describe possibilities for $\pi|_{O(n)}$ $\quad (\pi \in \widehat{GL(n,\mathbb{R})})$;
2. compute which possibility occurs for which π.

Big tools: algebraic geometry, commutative algebra.
Helps to change notation.

Thm. Cpt Lie group $K \rightsquigarrow$ complexification $K(\mathbb{C})$:

$$\text{cont reps of } K \simeq \text{alg reps of } K(\mathbb{C}).$$

New notation convenient for using $K(\mathbb{C})$:

old notation	new notation
$K = O(n)$	$K(\mathbb{R}) = O(n)$
$K(\mathbb{C}) = O(n,\mathbb{C})$	$K = O(n,\mathbb{C})$
$\mathfrak{g} = Lie(G) = \mathfrak{gl}(n,\mathbb{R})$	$\mathfrak{g}(\mathbb{R}) = \mathfrak{gl}(n,\mathbb{R})$
$\mathfrak{g}(\mathbb{C}) = Lie(G) \otimes_{\mathbb{R}} \mathbb{C}$	$\mathfrak{g} = \mathfrak{gl}(n,\mathbb{C})$

All works for any real reductive group with cplxified Lie alg $\mathfrak{g}$, cplxified max cpt K.

New notation suggests new questions

Old interest: $\mathcal{H}_\pi =$ irr unitary of $GL(n, \mathbb{R})$.

New interest: $V = \mathcal{H}_\pi^{K,\infty} = O(n, \mathbb{C})$-finite vecs.

$(\mathfrak{g}, K)$-module is vector space V with

1. alg repn π_K of algebraic group $K = O(n, \mathbb{C})$: $V = \sum_{\mu \in \widehat{K}} m_V(\mu)\mu$
2. repn $\pi_{\mathfrak{g}}$ of cplx Lie algebra $\mathfrak{g}$
3. $d\pi_K = \pi_{\mathfrak{g}}|_{\mathfrak{k}}, \qquad \pi_K(k)\pi_{\mathfrak{g}}(X)\pi_K(k^{-1}) = \pi_{\mathfrak{g}}(\mathrm{Ad}(k)X)$.

In module notation, cond (3) reads $k \cdot (X \cdot v) = (\mathrm{Ad}(k)X) \cdot (k \cdot v)$.

Two new goals today:

1. describe possibilities for $V|_K$;
2. compute $V|_K$ in interesting terms.

Bad answer: $m_V(\mu) =$ (formula with signs and partition fns).

Good answer: $V|_K \simeq$ (alg fns on variety with K action).

Finding varieties with K action

$O(n,\mathbb{C}) = K$ reductive alg gp $\curvearrowright \mathfrak{gl}(n,\mathbb{C}) = \mathfrak{g}$ cplx reduc Lie alg.

$\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$ skew symm $\oplus$ symm matrices

$\mathcal{N}^* =$ cone of nilp elts in $\mathfrak{g}^*$ cplx nilp matrices.

$\mathcal{N}^*_\theta = \mathcal{N}^* \cap \mathfrak{s}^*$, nilpotent symmetric matrices

$\mathcal{N}^*_\theta =$ finite # nilpotent K orbits $\mathcal{O}$.

[Irr $(\mathfrak{g}, K)$-mod V] $|_K \approx$ alg fns on some $\overline{\mathcal{O}}$.

In this language, our goals are

1. Attach nilp orbits to $(\mathfrak{g}, K)$-mods in theory.
2. Compute them in practice.

"In theory there is no difference between theory and practice. In practice there is." Jan L. A. van de Snepscheut (or not).

Classical limits for representations

Rep of $\mathfrak{g}$ is module for noncomm $U(\mathfrak{g})$: QUANTUM.

CLASSICAL ANALOGUE is module for comm $S(\mathfrak{g})$.

Fundamental link is PBW:

$$U(\mathfrak{g}) = \cup_{n\geq 0} U_n(\mathfrak{g}), \qquad U_p \cdot U_q \subset U_{p+q}$$

$$\text{gr}\, U(\mathfrak{g}) =_{\text{def}} \sum_{n\geq 0} U_n/U_{n-1}, \qquad \text{gr}\, U(\mathfrak{g}) \simeq S(\mathfrak{g}).$$

V fin gen/$U(\mathfrak{g})$, V_0 fin diml generating; set

$$V_n = U_n(\mathfrak{g}) \cdot V_0, \qquad \text{gr}\, V =_{\text{def}} \sum_{n\geq 0} V_n/V_{n-1}$$

finitely generated graded $S(\mathfrak{g})$-module.

V $(\mathfrak{g}, K)$-module, V_0 K-stable $\rightsquigarrow$ gr V $(S(\mathfrak{g}/\mathfrak{k}), K)$-module.

$V|_K \simeq (\text{gr}\, V)|_K$: res to K lives in classical world.

Thm. If V finite length $(\mathfrak{g}, K)$-module, then $(S(\mathfrak{g}/\mathfrak{k}), K)$-module gr V supported on $\mathcal{N}^*_\theta \subset \mathfrak{s}^*$.

Associated varieties

$\mathcal{F}(\mathfrak{g}, K)$ = finite length $(\mathfrak{g}, K)$-modules...

noncommutative world we care about.

$\mathcal{C}(\mathfrak{g}, K) =$ f.g. $(S(\mathfrak{g}/\mathfrak{k}), K)$-modules, support $\subset \mathcal{N}_\theta^* \ldots$

commutative world where geometry can help.

$$\mathcal{F}(\mathfrak{g}, K) \overset{\text{gr}}{\rightsquigarrow} \mathcal{C}(\mathfrak{g}, K)$$

Prop. gr induces surjection of Grothendieck groups

$$K\mathcal{F}(\mathfrak{g}, K) \xrightarrow{\text{gr}} K\mathcal{C}(\mathfrak{g}, K);$$

image records restriction to K of HC module.

So restrictions to K of HC modules sit in equivariant coherent sheaves on nilpotent cone in $(\mathfrak{g}/\mathfrak{k})^*$

$$K\mathcal{C}(\mathfrak{g}, K) =_{\text{def}} K^K(\mathcal{N}_\theta^*),$$

equivariant K-theory of the K-nilpotent cone.

Goal 2: compute $K^K(\mathcal{N}_\theta^*)$ and the map **Prop.**

Equivariant K-theory

Setting: (complex) algebraic group K acts on (complex) algebraic variety X.

$\mathrm{Coh}^K(X) =$ abelian categ of coh sheaves on X with K action.

$K^K(X) =_{\mathrm{def}}$ Grothendieck group of $\mathrm{Coh}^K(X)$.

Example: $\mathrm{Coh}^K(\mathrm{pt}) = \mathrm{Rep}(K)$ (fin-diml reps of K).

$K^K(\mathrm{pt}) = R(K) =$ rep ring of K; free $\mathbb{Z}$-module, basis $\widehat{K}$.

Example: $X = K/H$; $\mathrm{Coh}^K(K/H) \simeq \mathrm{Rep}(H)$

$E \in \mathrm{Rep}(H) \rightsquigarrow \mathcal{E} =_{\mathrm{def}} K \times_H E$ eqvt vector bdle on K/H

$K^K(K/H) = R(H)$.

Example: $X = V$ vector space (repn of K).

$E \in \mathrm{Rep}(K) \rightsquigarrow$ proj module
$\mathcal{O}_V(E) =_{\mathrm{def}} \mathcal{O}_V \otimes E \in \mathrm{Coh}^K(X)$

proj resolutions $\implies K^K(V) \simeq R(K)$, basis $\{\mathcal{O}_V(\tau)\}$.

Doing nothing carefully

Suppose $K \curvearrowright X$ with finitely many orbits:

$X = Y_1 \cup \cdots \cup Y_r, \qquad Y_i = K \cdot y_i \simeq K/K^{y_i}.$

Orbits partially ordered by $Y_i \geq Y_j$ if $Y_j \subset \overline{Y_i}$.

$(\tau, E) \in \widehat{K^{y_i}} \rightsquigarrow \mathcal{E}(\tau) \in \mathrm{Coh}^K(Y_i)$.

Choose (always possible) K-eqvt coherent extension

$$\widetilde{\mathcal{E}}(\tau) \in \mathrm{Coh}^K(\overline{Y_i}) \rightsquigarrow [\widetilde{\mathcal{E}}] \in K^K(\overline{Y_i}).$$

Class $[\widetilde{\mathcal{E}}]$ on $\overline{Y}_i$ unique modulo $K^K(\partial Y_i)$.

Set of all $[\widetilde{\mathcal{E}}(\tau)]$ (as Y_i and τ vary) is basis of $K^K(X)$.

Suppose $M \in \mathrm{Coh}^K(X)$; write class of M in this basis

$$[M] = \sum_{i=1}^{r} \sum_{\tau \in \widehat{K^{y_i}}} n_\tau(M)[\widetilde{\mathcal{E}}(\tau)].$$

Maxl orbits in $\mathrm{Supp}(M)$ = maxl Y_i with some $n_\tau(M) \neq 0$.

Coeffs $n_\tau(M)$ on maxl Y_i ind of choices of exts $\widetilde{\mathcal{E}}(\tau)$.

Our story so far

We have found

1. homomorphism

   virt $G(\mathbb{R})$ reps $K\mathcal{F}(\mathfrak{g}, K) \xrightarrow{\text{gr}} K^K(\mathcal{N}^*_\theta)$ eqvt K-theory
2. geometric basis $\left\{[\widetilde{\mathcal{E}(\tau)}]\right\}$ for $K^K(\mathcal{N}^*_\theta)$, indexed by irr reps of isotropy gps
3. expression of $[\text{gr}(\pi)]$ in geom basis $\rightsquigarrow \mathcal{AC}(\pi)$.

Problem is computing such expressions...

Teaser for the next section: Kazhdan and Lusztig taught us how to express π using std reps $I(\gamma)$:

$$[\pi] = \sum_\gamma m_\gamma(\pi)[I(\gamma)], \qquad m_\gamma(\pi) \in \mathbb{Z}.$$

$\{[\text{gr}\, I(\gamma)]\}$ is another basis of $K^K(\mathcal{N}^*_\theta)$.

Last goal is compute chg of basis matrix: to write

$$[\widetilde{\mathcal{E}}(\tau)] = \sum_\gamma n_\gamma(\tau)[\text{gr}\, I(\gamma)].$$

The last goal (last slide of actual lecture)

Studying cone $\mathcal{N}^*_\theta =$ nilp lin functionals on $\mathfrak{g}/\mathfrak{k}$.

Found (for free) basis $\left\{[\widetilde{\mathcal{E}(\tau)}]\right\}$ for $K^K(\mathcal{N}^*_\theta)$, indexed by orbit K/K^i and irr rep τ of K^i.

Found (by rep theory) second basis $\{[\text{gr } I(\gamma)]\}$, indexed by (parameters for) std reps of $G(\mathbb{R})$.

To compute associated cycles, enough to write

$$[\text{gr } I(\gamma)] = \sum_{\text{orbits}} \sum_{\substack{\tau \text{ irr for} \\ \text{isotropy}}} N_\tau(\gamma)[\widetilde{\mathcal{E}}(\tau)].$$

Equivalent to compute inverse matrix

$$[\widetilde{\mathcal{E}}(\tau)] = \sum_\gamma n_\gamma(\tau)[\text{gr } I(\gamma)].$$

Need to relate

geom of nilp cone $\leftrightsquigarrow$ geom of std reps.

Use parabolic subgps and Springer resolution.

Introducing Springer

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ Cartan decomp, $\mathcal{N}_\theta^* \simeq \mathcal{N}_\theta =_{\text{def}} \mathcal{N} \cap \mathfrak{s}$ nilp cone in $\mathfrak{s}$.

Kostant-Rallis, Jacobson-Morozov: nilp $X \in \mathfrak{s} \rightsquigarrow Y \in \mathfrak{s}$, $H \in \mathfrak{k}$

$$[H, X] = 2X, \quad [H, Y] = -2Y, \quad [X, Y] = H,$$

$$\mathfrak{g}[k] = \mathfrak{k}[k] \oplus \mathfrak{s}[k] \qquad (\text{ad}(H) \text{ eigenspace}).$$

$\rightsquigarrow \mathfrak{g}[\geq 0] =_{\text{def}} \mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ θ-stable parabolic.

Theorem (Kostant-Rallis) Write $\mathcal{O} = K \cdot X \subset \mathcal{N}_\theta$.

1. $\mu\colon \mathcal{O}_Q =_{\text{def}} K \times_{Q \cap K} \mathfrak{s}[\geq 2] \to \overline{\mathcal{O}}$, $(k, Z) \mapsto \text{Ad}(k)Z$ is proper birational map onto $\overline{\mathcal{O}}$.
2. $K^X = (Q \cap K)^X = (L \cap K)^X (U \cap K)^X$ is a Levi decomp; so $\widehat{K^X} = [(L \cap K)^X]^\wedge$.

So have resolution of singularities of $\overline{\mathcal{O}}$:

$$\begin{array}{ccc} & K \times_{Q \cap K} \mathfrak{s}[\geq 2] & \\ \text{vec bdle} \swarrow & & \searrow \mu \\ K/Q \cap K & & \overline{\mathcal{O}} \end{array}$$

Use it (*i.e.*, copy McGovern, Achar) to calculate equivariant K-theory…

Using Springer to calculate K-theory

$X \in \mathcal{N}_\theta$ represents $\mathcal{O} = K \cdot X$.

$\mu\colon \mathcal{O}_Q =_{\text{def}} K \times_{Q\cap K} \mathfrak{s}[\geq 2] \to \overline{\mathcal{O}}$ Springer resolution.

Theorem Recall $\widehat{K^X} = [(L\cap K)^X]^\wedge$.

1. $K^K(\mathcal{O}_Q)$ has basis of eqvt vec bdles:
$$(\sigma, F) \in \text{Rep}(L \cap K) \rightsquigarrow \mathcal{F}(\sigma).$$
2. Get extension of $\mathcal{E}(\sigma|_{(L\cap K)^X})$ from $\mathcal{O}$ to $\overline{\mathcal{O}}$
$$[\overline{\mathcal{F}}(\sigma)] =_{\text{def}} \sum_i (-1)^i [R^i \mu_*(\mathcal{F}(\sigma))] \in K^K(\overline{\mathcal{O}}).$$
3. Compute (very easily) $[\overline{\mathcal{F}}(\sigma)] = \sum_\gamma n_\gamma(\sigma)[\text{gr}\, I(\gamma)]$.
4. Each irr $\tau \in [(L\cap K)^X]^\wedge$ extends to (virtual) rep $\sigma(\tau)$ of $L \cap K$; can choose $\overline{\mathcal{F}(\sigma(\tau))}$ as extension of $\mathcal{E}(\tau)$.

Now we can compute associated cycles

Recall $X \in \mathcal{N}_\theta \rightsquigarrow \mathcal{O} = K \cdot X$; $\tau \in [(L \cap K)^X]\widehat{}\ $.

We now have explicitly computable formulas

$$[\widetilde{\mathcal{E}}(\tau)] = [\overline{\mathcal{F}(\sigma(\tau))}] = \sum_\gamma n_\gamma(\tau)[\mathrm{gr}\, I(\gamma)].$$

Here's why this does what we want:

1. inverting matrix $n_\gamma(\tau) \rightsquigarrow$ matrix $N_\tau(\gamma)$ writing $[\mathrm{gr}\, I(\gamma)]$ in terms of $[\widetilde{\mathcal{E}}(\tau)]$.
2. multiplying $N_\tau(\gamma)$ by Kazhdan-Lusztig matrix $m_\gamma(\pi)$ $\rightsquigarrow$ matrix $n_\tau(\pi)$ writing $[\mathrm{gr}\, \pi]$ in terms of $[\widetilde{\mathcal{E}}(\tau)]$.
3. Nonzero entries $n_\tau(\pi) \rightsquigarrow \mathcal{AC}(\pi)$.

Side benefit: algorithm for $G(\mathbb{R})$ cplx also computes a bijection (conj Lusztig, proof Bezrukavnikov)

(dom wts) $\leftrightarrow$ (pairs $(\mathcal{O}, \tau)$)...

Complex groups regarded as real

G_1 = cplx conn reductive alg gp $\leftrightsquigarrow$ old $G(\mathbb{R})$).

σ_1 = cplx conj for compact real form of G_1.

$G = G_1 \times G_1$ complexification of $G_1 \ldots$

1. $\sigma(x, y) = (\sigma_1(y), \sigma_1(x))$ cplx conj for real form G_1:
$$G(\mathbb{R}) = G^\sigma = \{(x, \sigma_1(x) \mid x \in G_1\} \simeq G_1.$$
2. $\theta(x, y) = (y, x)$ Cartan inv: $K = G^\theta = (G_1)_\Delta$.

K-nilp cone $\mathcal{N}_\theta^* \subset \mathfrak{g}^* \simeq G_1$-nilp cone $\mathcal{N}_1^* \subset \mathfrak{g}_1^*$.

$H_1 \subset G_1$, $H = H_1 \times H_1 \subset G$, $T = (H_1)_\Delta \subset K$ max tori.

$\mathfrak{a} = \mathfrak{h}^{-\theta} = \{(Z, -Z) \mid Z \in \mathfrak{h}_1\}$ Cartan subspace.

Param for princ series rep is $\gamma = (\lambda, \nu) \in X^*(T) \times \mathfrak{a}^*$:

1. $I(\lambda, \nu)|_K \simeq \mathrm{Ind}_T^K(\lambda)$;
2. virt rep $[I(w_1 \cdot \lambda, w_1 \cdot \nu)]$ indep of $w_1 \in W_1$;
3. $[\mathrm{gr}\, I(\lambda, \nu)] \in K^K(\mathcal{N}_\theta^*) \simeq K^{G_1}(\mathcal{N}_1^*)$ indep of ν.

Conclusion: the set of all $[\mathrm{gr}\, I(\lambda)] \simeq \mathrm{Ind}_T^K(\lambda)$ ($\lambda \in X^*(T)$ dom) is basis for (virt HC-mods of G_1)$|_K$.

Connection with Weyl char formula

$K \simeq G_1$ cplx conn reductive alg, $T \simeq H_1$ max torus.

Asserted "$\left\{\mathrm{Ind}_T^K(\lambda)\right\}$ basis for (virt HC-mods of G_1) $|_K$."

What's that mean or tell you?

Fix $(F,\mu) \in \widehat{K}$ of highest weight $\mu \in X^{\mathrm{dom}}(T)$.

(F,μ) also irr (fin diml) HC-mod for G_1; $(F,\mu)|_K = (F,\mu)$.

Assertion means $F = \sum_{\gamma \in X^{\mathrm{dom}}(T)} m_\gamma(F)\, \mathrm{Ind}_T^K(\gamma)$.

Such a formula is a version of Weyl char formula:

$$\begin{aligned}(F,\mu) &= \sum_{w \in W(K,T)} (-1)^{\ell(w)}\, \mathrm{Ind}_T^K(\mu + \rho - w\rho) \\ &= \sum_{B \subset \Delta^+(\mathfrak{k},\mathfrak{t})} (-1)^{|\Delta^+| - |B|}\, \mathrm{Ind}_T^K(\mu + 2\rho - 2\rho(B)).\end{aligned}$$

One meaning: if $(E,\gamma) \in \widehat{K}$, then

$$\sum_{w \in W} (-1)^{\ell(w)} m_{E,\gamma}(\mu + \rho - w \cdot \rho) = \begin{cases} 1 & (\gamma = \mu) \\ 0 & (\gamma \neq \mu). \end{cases}$$

Lusztig's conjecture

$G \supset B \supset H$ complex reductive algebraic.

$X^*(H) \supset X^{\text{dom}}(H)$ dominant weights.

$\mathcal{N}^* =$ cone of nilpotent elements in $\mathfrak{g}^*$.

Lusztig conjecture: there's a bijection

$$X^{\text{dom}} \leftrightsquigarrow \text{pairs } (\xi, \tau)/G \text{ conjugation;}$$

$\xi \in \mathcal{N}^*$, $\tau \in \widehat{G^\xi} \leftrightsquigarrow$ eqvt vec bdle $\mathcal{E}(\tau) = G \times_{G^\xi} \tau$

Thm (Bezrukavnikov). There is a preferred virt extension $\widetilde{\mathcal{E}}(\tau)$ to $\overline{G \cdot \xi}$ so

$$[\widetilde{\mathcal{E}}(\tau)] = \pm[\text{gr } I(\lambda(\xi,\tau))] + \sum_{\gamma \prec \lambda(\xi,\tau)} n_\gamma(\xi,\tau)[\text{gr } I(\gamma)].$$

Upper triangularity characterizes Lusztig bijection.

Calculating Lusztig's bijection

Proceed by upward induction on nilpotent orbit.

Start with (ξ,τ), $\xi \in \mathcal{N}^*$, $\tau \in \widehat{G^\xi}$.

JM parabolic $Q = LU$, $\xi \in (\mathfrak{g}/\mathfrak{q})^*$; $G^\xi = Q^\xi = L^\xi U^\xi$.

Choose virt rep $[\sigma(\tau)] \in R(L)$ extension of τ.

Write formula for corr ext of $\mathcal{E}(\tau)$ to $\overline{G\cdot\xi}$:

$$[\overline{\mathcal{F}(\sigma(\tau))}] = \sum_\lambda m_{\sigma(\tau)}(\lambda) \sum_{B\subset\Delta^+(\mathfrak{l},\mathfrak{h})} (-1)^{|\Delta^+(\mathfrak{l},\mathfrak{h})|-|B|} \sum_{A\subset\Delta(\mathfrak{g}[1],\mathfrak{h})} (-1)^{|A|}$$

$$[\mathrm{gr}\, I(\lambda + 2\rho_L - 2\rho(A) - 2\rho(B))].$$

Rewrite with $[\mathrm{gr}\, I(\lambda')]$, λ' dominant.

Loop: find largest λ'.

If $\lambda' \leftrightsquigarrow (\xi',\tau')$ for smaller $G\cdot\xi'$, subtract

$$m_{\sigma(\tau)}(\lambda') \times \text{formula for } (\xi',\tau');$$

$\rightsquigarrow$ new formula for (ξ,τ) with smaller leading term.

When loop ends, $\lambda' = \lambda(\xi,\tau)$.

What to do next

Sketched effective algorithms for computing assoc cycles for HC modules, Lusztig bijection.

What should we (this means you) do now?

Software implementations of these?

- Pramod Achar $\rightsquigarrow$ `gap` script for Lusztig bij in type A.
- Marc van Leeuwen $\rightsquigarrow$ `atlas` software for (std rep)$|_K$.

Real group version of Lusztig bijection?

- Algorithm still works, but bijection not quite true.
- Failure partitions $\widehat{K}$ into small finite sets.

Closed form information about algorithms?

- formula for smallest $\lambda \leftrightsquigarrow$ (one orbit, any τ);
- Would bound below infl char of HC-mod $\leftrightsquigarrow$ orbit.