THREE-DIMENSIONAL SUBGROUPS AND UNITARY REPRESENTATIONS

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The simplest noncommutative compact Lie group is the group $SU(2)$ of unit quaternions. If K is a compact Lie group, write $D(K)$ for the set of conjugacy classes of homomorphisms of $SU(2)$ into K. Dynkin showed in the 1950s that $D(K)$ is a finite set, and calculated it in all cases.

A fundamental unsolved problem is to parametrize the "purely real" unramified unitary representations of a split reductive group G over a local field. Such representations are parametrized by a compact polytope $P(G)$. When G and K are "Langlands dual" to each other, a conjecture of Arthur realizes $D(K)$ as a subset of $P(G)$. We discuss the status of this conjecture, and how Dynkin's problem illuminates the representation-theoretic one.

1 Introduction

One of the purposes of representation theory is to provide tools for harmonic analysis problems. The idea is to understand actions of groups on geometric objects by understanding first the possible representations of the group (by linear operators). Formally the simplest examples are finite groups: no sophisticated analytical tools are needed to study them. Nevertheless the (finite set) of irreducible representations of a finite group can be extraordinarily complicated from a combinatorial point of view. In some respects the representation theory of (connected) Lie groups is actually simpler than that of finite groups, because the geometric structure of a Lie group constrains the multiplication law to be nearly commutative.

The purpose of this paper is to examine a classical problem in the representation theory of Lie groups (formulated as (23) below). The problem is still unsolved. I'll explain a conjecture due to James Arthur that relates this representation theory problem to a structural problem for compact groups. The structural problem was solved by Eugene Dynkin in the 1950s. Connecting the two problems requires the classification of compact Lie groups in the beautiful form given to it by Michel Demazure and Alexandre Grothendieck (elaborating on previous constructions). I will recall that classification in Sec. 2. The solution to Dynkin's problem appears in Sec. 3. In Sec. 4 I will formulate the representation-theoretic problem, and state Arthur's conjecture about it (see (24) below). Finally, Sec. 5 outlines results of Barbasch and Moy proving (24) for p-adic fields.

2 Compact Lie groups

The formulation of Dynkin's problem can be motivated by a fundamental idea from topology. There one begins with the spheres S^n as the most basic examples of topological spaces. One can then study a general space X by studying the (continuous) maps from spheres to X. The *nth homotopy group of* X is

$$\pi_n(X) = \text{maps}(S^n, X)/\text{homotopy}.$$

In the category of compact Lie groups, there are two simplest examples: the unit circle in the multiplicative group of complex numbers, and the unit 3-sphere in the quaternions. That is,

$$\begin{aligned} S^1 &\simeq \ U(1) \ \simeq \text{unit sphere in } \mathbb{C} \\ S^3 &\simeq SU(2) \simeq \text{unit sphere in } \mathbb{H} \end{aligned} \tag{1}$$

In analogy with the homotopy groups of topological spaces, it is then natural to attach to a compact Lie group K two invariants:

$$\begin{aligned} C(K) &= (\text{homomorphisms from } U(1) \text{ to } K)/(\text{conjugation by } K) \\ D(K) &= (\text{homomorphisms from } SU(2) \text{ to } K)/(\text{conjugation by } K) \end{aligned} \tag{2}$$

A description of the invariant $C(K)$ is implicit in the classical structure theory for compact Lie groups, which we will recall below. This set may be naturally identified with the orbits of a finite group on a lattice. The invariant $D(K)$, which is a finite set, was computed by Dynkin;[10] we will discuss his result in Sec. 3.

In order to understand the invariant $C(K)$, we begin with the special case when K is a compact torus. It is convenient and traditional in this case to use a slightly different notation. Suppose therefore that T is a compact connected abelian Lie group. The *lattice of one-parameter subgroups of* T is

$$X_*(T) = \text{continuous homomorphisms from } U(1) \text{ to } T. \tag{3}$$

Dually, the *lattice of weights of* T is

$$X^*(T) = \text{continuous homomorphisms from } T \text{ to } U(1). \tag{4}$$

Because of the duality between these notions, one-parameter subgroups are also called *coweights.*

Theorem 1 *Suppose T is a compact connected abelian Lie group. Then the sets $X_*(T)$ and $X^*(T)$ defined in (3) and (4) are naturally lattices (that is, free abelian groups) of rank equal to the dimension of T. There is a natural pairing $\langle , \rangle$ from $X_*(T) \times X^*(T)$ into $\mathbb{Z}$ identifying each lattice as the dual of the other.*

The set $C(T)$ defined in (2) may be identified with the lattice $X_*(T)$.

We omit the elementary proof, but recall the construction of the pairing. Suppose $\gamma: U(1) \to T$ is a one-parameter subgroup and $\lambda: T \to U(1)$ is a weight. Then the composition $\lambda \circ \gamma$ is a homomorphism from $U(1)$ to itself, which therefore sends $e^{i\theta}$ to $e^{im\theta}$ for some integer m. We define $\langle \gamma, \lambda \rangle = m$.

We return now to the setting of a compact connected Lie group K. Recall that a *maximal torus in K* is a connected abelian Lie subgroup $T \subset K$ of maximal dimension. The *Weyl group of T in K* is equal to the normalizer of T in K modulo T:

$$W = W(K,T) = N_K(T)/T. \tag{5}$$

It is a finite group of automorphisms of the torus T, and therefore acts also by automorphisms on the lattices $X_*(T)$ and $X^*(T)$.

Theorem 2 *Suppose K is a compact connected Lie group. Any two maximal tori T and T' in K are conjugate by K. This conjugation defines isomorphisms*

$$X_*(T) \simeq X_*(T'), \qquad X^*(T) \simeq X^*(T')$$

which are uniquely determined up to composition with an automorphism from $W(K,T)$.

Any homomorphism from $U(1)$ to K is conjugate to one mapping to T. Two homomorphisms from $U(1)$ to T are conjugate by K if and only if they are conjugate by $W(K,T)$. Consequently the set $C(K)$ defined in (2) may be identified with the set of orbits of $W(K,T)$ on the lattice $X_*(T)$:

$$C(K) \simeq X_*(T)/W(K,T).$$

Proofs may be found in many texts, including Knapp.[13]

This is all the structure theory for compact groups that we need to describe Dynkin's determination of $D(K)$. In order to make the connection to our representation-theoretic problem, we will need a bit about root systems. For proofs and more details, the reader may again consult Knapp.[13]

We continue to fix a maximal torus T in a compact connected Lie group K. Write

$$\mathfrak{k} = \mathrm{Lie}(K), \qquad \mathfrak{k}_{\mathbb{C}} = \mathfrak{k} \otimes_{\mathbb{R}} \mathbb{C}$$

for the Lie algebra of K and its complexification. We use similar notation for T. The group K acts on $\mathfrak{k}_{\mathbb{C}}$ by Lie algebra automorphisms; this defines a complex representation of K, called Ad. The restriction of Ad to T splits (like any finite-dimensional complex representation of T) into a direct sum of one-dimensional representations. The action of T on $\mathfrak{t}_{\mathbb{C}}$ is trivial (since T is abelian). All the other characters of T appearing in Ad are non-trivial, and each appears exactly once. We find in this way a decomposition

$$\mathfrak{k}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \sum_{\alpha \in R(K,T)} \mathfrak{k}_{\mathbb{C}}^{\alpha}. \tag{6}$$

Here $R(K,T) \subset X^*(T)$ is the set of non-trivial weights of $\mathrm{Ad}(T)$ acting on $\mathfrak{k}_{\mathbb{C}}$. It is called *the set of roots of T in K*. The one-dimensional subspace $\mathfrak{k}_{\mathbb{C}}^{\alpha}$ is called a *root space*. Clearly

$$|R(K,T)| = \dim \mathfrak{k}/\mathfrak{t} = \dim K/T,$$

so that in particular the root system is a finite subset of $X^*(T)$.

The traditional approach to the classification of compact Lie groups is based on the geometry of the root system, which is characterized by fairly simple axioms (see for example Knapp,[13] section II.5). Our purposes (including the connection with representation theory) are better served by the Demazure-Grothendieck notion of root datum, which requires one more idea. In order to express it, notice first that the compact group $SU(2)$ has a maximal torus T_1 consisting of diagonal matrices, which is naturally identified with $U(1)$:

$$T_1 = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \simeq U(1) \tag{7}$$

Next, the complexified Lie algebra of $SU(2)$ is naturally identified with two by two complex matrices of trace 0. The two root subspaces are

$$\begin{aligned} \mathfrak{su}(2)_{\mathbb{C}}^{\alpha_1} &= \left\{ \left(\begin{smallmatrix} 0 & z \\ 0 & 0 \end{smallmatrix}\right) \mid z \in \mathbb{C} \right\}, \\ \mathfrak{su}(2)_{\mathbb{C}}^{-\alpha_1} &= \left\{ \left(\begin{smallmatrix} 0 & 0 \\ w & 0 \end{smallmatrix}\right) \mid w \in \mathbb{C} \right\}, \end{aligned} \tag{8}$$

The identification of T_1 with $U(1)$ identifies the lattices $X_*(T_1)$ and $X^*(T_1)$ with $\mathbb{Z}$. In this identification, the root α_1 is identified with 2. (This means that $\mathrm{Ad}\left(\begin{smallmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{smallmatrix}\right)$ acts on $\mathfrak{k}_{\mathbb{C}}^{\alpha_1}$ by multiplication by $e^{2i\theta}$.)

Theorem 3 *Suppose K is a compact connected Lie group, T is a maximal torus in K, and $\alpha \in R(K,T)$ is a root. Then there is a group homomorphism*

$$\phi_\alpha : SU(2) \to K,$$

satisfying $\phi_\alpha(T_1) \subset T$, and

$$d\phi_\alpha(\mathfrak{su}(2)^{\alpha_1}_{\mathbb{C}}) = \mathfrak{k}^{\alpha}_{\mathbb{C}}.$$

The homomorphism ϕ_α is uniquely determined up to conjugation by T in K or by T_1 in $SU(2)$. In particular, the restriction of ϕ_α to $T_1 \simeq U(1)$ is a well-defined homomorphism from $U(1)$ to T. This restriction determines α uniquely.

We omit the elementary proof; most of the necessary ingredients may be found on page 209 of Knapp.[13]

The restriction of ϕ_α to T_1 described in the proposition is called *the coroot corresponding to* α, and written

$$\alpha^\vee = \phi_\alpha \,|_{T_1} \in X_*(T). \tag{9}$$

The set of all coroots of T in K is written $R^\vee(K,T) \subset X_*(T)$. According to Theorem 3, the map $\alpha \mapsto \alpha^\vee$ is a bijection from $R(K,T)$ to $R^\vee(K,T)$.

In order to understand the notion of root datum, we need to know how roots and coroots are related to the Weyl group $W(K,T)$ defined in (5). Here is the result.

Theorem 4 *Suppose K is a compact connected Lie group, T is a maximal torus in K, and $\alpha \in R(K,T)$ is a root. Fix a homomorphism ϕ_α as in Theorem 3, and define*

$$\sigma_\alpha = \phi_\alpha \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then $\sigma_\alpha \in N_K(T)$. The coset $\sigma_\alpha T$ is independent of the choice of ϕ_α, and therefore gives a well-defined element $s_\alpha \in W(K,T)$. The action of s_α on T is given by

$$s_\alpha(t) = t \cdot \left(\alpha^\vee(\alpha(t))^{-1}\right).$$

The actions on one-parameter subgroups and on weights are

$$s_\alpha(\gamma) = \gamma - \langle \gamma, \alpha \rangle \alpha^\vee \qquad (\gamma \in X_*(T)),$$

$$s_\alpha(\lambda) = \lambda - \langle \alpha^\vee, \lambda \rangle \alpha \qquad (\lambda \in X^*(T)).$$

The elements s_α have order 2, and they generate $W(K,T)$.

Everything except the assertion that the elements s_α generate $W(K,T)$ is elementary. Again, a proof may be assembled from page 209 of Knapp.[13] Given the formulas for s_α, the fact that s_α has order two is equivalent to $\langle \alpha^\vee, \alpha \rangle = 2$. This fact in turn comes down to the identification of the root α_1 of $SU(2)$ with 2, explained after (8). The element s_α is called *the reflection in the root* α.

Finally we can give the definition of a root datum due to Demazure and Grothendieck. For more details we refer to Springer.[19] A *root datum* is a quadruple $\Psi = (X, R, X^\vee, R^\vee)$. Here X and $X^\vee$ are lattices (that is, free abelian groups of finite rank) dual to each other by a fixed pairing $\langle , \rangle$ mapping $X^\vee \times X$ to $\mathbb{Z}$. The sets R and $R^\vee$ are finite subsets of X and $X^\vee$ respectively, endowed with a fixed bijection $\alpha \mapsto \alpha^\vee$ from R to $R^\vee$. For each $\alpha \in R$ we define endomorphisms s_α of X and $X^\vee$ by

$$s_\alpha(\lambda) = \lambda - \langle \alpha^\vee, \lambda \rangle \alpha, \qquad s_\alpha(\gamma) = \gamma - \langle \gamma, \alpha \rangle \alpha^\vee \qquad (\lambda \in X, \gamma \in X^\vee).$$

This structure is subject to just two axioms:

(RD1) If $\alpha \in R$, then $\langle \alpha^\vee, \alpha \rangle = 2$.

(RD2) If $\alpha \in R$, then $s_\alpha(R) \subset R$, and $s_\alpha(R^\vee) \subset R^\vee$.

The *Weyl group of the root datum* is by definition the group $W(\Psi)$ of automorphisms of X generated by the s_α. It is naturally isomorphic (by an inverse transpose map) to the corresponding group of automorphisms of $X^\vee$.

The root datum is called *reduced* if it satisfies also the axiom

(RD0) If $\alpha \in R$, then $2\alpha \notin R$ and $2\alpha^\vee \notin R^\vee$.

Because of the symmetry of the axioms, it is immediately obvious that to every root datum $\Psi = (X, R, X^\vee, R^\vee)$ is associated the *dual root datum* $\Psi^\vee = (X^\vee, R^\vee, X, R)$. It is reduced exactly when Ψ is. Because of Theorem 4, it is easy to see that if K is a compact Lie group with maximal torus K, then $\Psi(K,T) = (X^*(T), R(K,T), X_*(T), R^\vee(K,T))$ is a root datum, called *root datum of* T *in* K; its Weyl group is the Weyl group of T in K. It is a standard fact about compact groups that this root datum is reduced. By Theorem 2, any two root data for K are isomorphic, and the isomorphism is canonically defined up to composition with an element of the Weyl group.

Theorem 5 *Suppose* (K,T) *and* $((K', T')$ *are pairs consisting of a compact connected Lie group and a maximal torus. Suppose that the corresponding root data* $\Psi(K,T) = (X^*(T), R(K,T), X_*(T), R^\vee(K,T))$ *and* $\Psi(K',T') = (X^*(T'), R(K',T'), X_*(T'), R^\vee(K',T'))$ *are isomorphic by an isomorphism* j^*

from $X^*(T)$ *to* $X^*(T')$ *(and its inverse transpose* $j_*: X_*(T) \to X_*(T')$*). Then there is a group isomorphism* j *from* K *to* K' *carrying* T *to* T'*, and inducing the isomorphisms* j^* *and* j_* *in the obvious sense. Any two such group isomorphisms differ by composition with an inner automorphism from* T *(or* T'*).*

Finally, suppose Ψ *is any reduced root datum. Then there is a compact connected Lie group* K *and a maximal torus* $T \subset K$ *so that* $\Psi(K,T) \simeq \Psi$*.*

Unfortunately, I do not know a good reference for this result. The analogue for algebraic groups over an algebraically closed field is in SGA 3.[8] There is a close relationship, implemented by a "complexification" functor, between compact Lie groups and complex reductive algebraic groups. On the Lie algebra level this relationship can be found in many places (for example Knapp,[13] Theorem 6.11); but the relationship of the groups is harder to find. In any case, this relationship reduces Theorem 5 to the complex case treated in SGA3.[8]

Theorem 5 reduces the classification of compact Lie groups to the classification of root data. This in turn is closely related to the classification of root systems. We do not need this classification, so we will not discuss it further.

We can now begin to understand the role of root data and their duality in representation theory. If T is a compact torus (with root datum $(X_*(T), \emptyset, X^*(T), \emptyset)$) then the dual root datum corresponds to a compact torus $T^\vee$, called the *dual torus of* T. The weights of T are in one-to-one correspondence with the one-parameter subgroups of $T^\vee$. More generally, if K is a compact connected Lie group with root datum $\Psi = \Psi(K,T)$, then Theorem 5 guarantees that $\Psi^\vee$ is the root datum of a *dual compact Lie group* $K^\vee$ with maximal torus $T^\vee$. (The notation is consistent: $T^\vee$ is actually the dual torus to T.) The Cartan-Weyl highest weight theory says that irreducible representations of K are in one-to-one correspondence with $W(K,T)$ orbits on the lattice $X^*(T)$ of weights of T. The duality provides an isomorphism of Weyl groups $W(K,T) \simeq W(K^\vee, T^\vee)$, respecting the actions of these groups on $X^*(T) \simeq X_*(T)$. In light of Theorem 2, the conclusion is that irreducible representations of K are in one-to-one correspondence with $K^\vee$-conjugacy classes of one-parameter subgroups of the dual group. In the notation of (2),

$$\widehat{K} \simeq X^*(T)/W(K,T) \simeq X_*(T^\vee)/W(K^\vee, T^\vee) \simeq C(K^\vee). \tag{10}$$

Roughly speaking, information about representations of K is encoded by information about elements (precisely, one-parameter subgroups) of the dual group.

A central theme of the Langlands program[15] is enormous generalizations of (10). The compact group K can be replaced first by a reductive group over a local field, and later even by an adele group over a number field. In the latter setting the irreducible representations of K are replaced by spaces of automorphic forms. The compact dual group is replaced by the (closely related) L-group. Instead of one-parameter subgroups of the dual group, one considers various kinds of homomorphisms of Galois groups (and related groups) into the L-group. (I learned from Mark Reeder the beautiful idea that these homomorphisms can be thought of as "arithmetic elements" of the L-group. Then the (local) Langlands philosophy says that irreducible representations of a reductive group over a local field are (approximately) parametrized by arithmetic elements of the L-group.)

Langlands' original conjectures have been refined and clarified in many ways, but still have been proved only in a few cases. Our concern here will be with a very special case of conjectures of Arthur, concerning the role of unipotent elements in the L-group. The Jacobson-Morozov theorem says that such elements may be parametrized (up to conjugacy) by homomorphisms of $SU(2)$ into the compact dual group; that is (in the notation of (2)) by $D(K^\vee)$. We will explain Dynkin's computation of $D(K^\vee)$ in Sec. 3, and then formulate Arthur's conjecture precisely in Sec. 4.

Here is a basic example. Suppose K is the group $U(n)$ of $n \times n$ unitary matrices. As a maximal torus in K we can choose the group T of diagonal matrices. There is a natural isomorphism $T \simeq U(1)^n$ given by the diagonal entries. In these coordinates, we find

$$X_*(T) \simeq \mathbb{Z}^n, \qquad X^*(T) \simeq \mathbb{Z}^n. \tag{11}$$

For example, the one-parameter subgroup $\gamma(m_1, \ldots, m_n)$ corresponding to an n-tuple of integers sends $e^{i\theta}$ to the diagonal matrix with diagonal entries $e^{im_1\theta}, \ldots, e^{im_n\theta}$. The natural pairing between $X_*(T)$ and $X^*(T)$ is given by the usual inner product. Both roots and coroots correspond to ordered pairs (i,j) of distinct integers between 1 and n: in terms of the standard basis $\{e_1, \ldots, e_n\}$ of $\mathbb{Z}^n$, $\alpha_{(i,j)} = e_i - e_j$ (and similarly for $\alpha^\vee_{(i,j)}$). From the formula in Theorem 4, it is easy to check that the reflection $s_{\alpha_{(i,j)}}$ interchanges the i and j coordinates. Consequently the Weyl group is the group of all permutations of the coordinates:

$$W(K,T) \simeq S_n.$$

Notice that the root datum of $U(n)$ is isomorphic to its own dual.

We conclude this section with one more kind of structure on a root datum, which will appear repeatedly in the following sections. Suppose that $\Psi = (X, R, X^\vee, R^\vee)$ is a root datum, with Weyl group $W(\Psi)$. Because $s_\alpha(\alpha) = -\alpha$ for every root α, the roots R consist of pairs $\{\alpha, -\alpha\}$ of non-zero elements. Similarly, the coroots occur in pairs $\{\alpha^\vee, -\alpha^\vee\}$. A *positive system for* Ψ is a good way of picking out one root or coroot from each such pair. Here is a precise definition. A coweight $\gamma \in X^\vee$ is called *regular* if $\langle \gamma, \alpha \rangle \neq 0$ for all $\alpha \in R$; it is called *singular* if it is not regular. The singular elements of $X^\vee$ are evidently the union of at most $|R|$ proper subgroups of $X^\vee$. It follows that there exist regular elements of $X^\vee$. A *set of positive roots for* Ψ is a subset $R^+ \subset R$ so that there is a regular element $\gamma \in X^\vee$ with

$$R^+ = \{\alpha \in R \mid \langle \gamma, \alpha \rangle > 0\}. \tag{12}$$

As immediate consequences of this definition, we get

$$R = R^+ \cup -R^+ \qquad \text{(disjoint union)}$$

$$(R^+ + R^+) \cap R \subset R^+.$$

A positive root that is *not* the sum of two other positive roots is called *simple.* The collection of simple roots in R^+ is written $\Pi(R^+)$. A coweight $\gamma' \in X^\vee$ is called *dominant for* R^+ if

$$\langle \gamma', \alpha \rangle \geq 0, \qquad \text{all } \alpha \in R^+.$$

All of these definitions can be made in exactly the same way for coroots.

Theorem 6 *Suppose $\Psi = (X, R, X^\vee, R^\vee)$ is a root datum, and R^+ is a system of positive roots for Ψ.*

1. *The set of coroots $(R^+)^\vee = \{\alpha^\vee \mid \alpha \in R^+\}$ is a set of positive coroots for Ψ.*

2. *The set $\Pi(R^+)$ is a basis for the lattice generated by R. Every positive root is a non-negative combination of these basis elements.*

3. *Every coweight $\gamma' \in X^\vee$ is conjugate by $W(\Psi)$ to a unique dominant coweight; and every weight $\lambda \in X$ is conjugate by $W(\Psi)$ to a unique dominant weight.*

4. *Any two positive root systems for Ψ are conjugate by a unique element of $W(\Psi)$.*

Here is how these definitions look in the example of $U(n)$ (see (11) above). An n-tuple of integers $\gamma = (\gamma_1, \ldots, \gamma_n)$ is regular exactly when all the integers are distinct. The corresponding set of positive roots is

$$R^+ = \{e_i - e_j \mid \gamma_i > \gamma_j\}.$$

That is, positive root systems correspond precisely to orderings of the n coordinates; so the Weyl group S_n acts simply transitively on them. A coweight γ' is dominant if and only if its coordinates decrease in the specified ordering.

3 Dynkin's classification

We turn now to Dynkin's description of the homomorphisms from $SU(2)$ to a compact connected Lie group K. Theorem 3 already describes some such homomorphisms, characterizing them by their restrictions to the maximal torus T_1 of $SU(2)$. Dynkin's first result extends a part of this characterization to general homomorphisms of $SU(2)$ into K.

Theorem 7 *Suppose K is a compact connected Lie group, T is a maximal torus in K, and*

$$\phi: SU(2) \to K$$

is a group homomorphism. Recall that $T_1 \simeq U(1)$ is the standard maximal torus in $SU(2)$. Then there is a K-conjugate ϕ' of ϕ with the property that

$$\phi'(T_1) \subset T.$$

The homomorphism $\phi'|_{T_1}$ is uniquely determined by ϕ up to conjugation by $N_K(T)$. In particular, the one-parameter subgroup

$$\gamma(\phi) = \phi'|_{T_1}: U(1) \to T \in X_*(T)$$

is determined by ϕ up to the action of the Weyl group $W(K,T)$.

The one-parameter subgroup $\gamma(\phi)$ must belong to the sublattice of $X_*(T)$ generated by the coroots of T in K.

Conversely, suppose that $\gamma: U(1) \to T$ is a one-parameter subgroup of T. Then there is at most one K-conjugacy class of homomorphisms $\phi: SU(2) \to K$ with $\gamma(\phi) = \gamma$.

The analogue of this theorem for complex algebraic groups is due to Mal'cev;[16] a proof of that version may be found in Theorem 3.4.12 of Collingwood-McGovern.[6] It is not difficult to modify that proof to produce the result stated here. (The middle assertion, that $\gamma(\phi)$ lies in the coroot lattice, may require some comment. First, the group $SU(2)$ is equal to its own commutator subgroup. Its image under ϕ' must therefore be contained in the commutator subgroup of K. The one-parameter subgroups of T taking values in the commutator subgroup are exactly the rational combinations of the coroots. So $\gamma(\phi)$ must be a rational combination of coroots. Because $SU(2)$ is simply connected, the homomorphism ϕ' lifts to any covering group of K. Essentially because of Theorem 5, there is such a covering group for which the only coweights that are rational combinations of coroots are the integer combinations of coroots.)

Theorem 7 makes it reasonable to define

$$D(R^\vee) = \{\gamma \in X_*(T) \mid \gamma = \phi|_{T_1},\ \text{some } \phi\colon SU(2) \to K\}, \qquad (13)$$

the Dynkin coweights for $R^\vee$. Obviously $D(R^\vee)$ is invariant under the Weyl group $W(K,T)$. Theorem 7 identifies the invariant $D(K)$ (of conjugacy classes of homomorphisms of $SU(2)$ into K) with $W(K,T)$ orbits on $D(R^\vee)$. Our next task is to get some *a priori* information about $D(R^\vee)$. For that purpose, we fix a set of positive roots $R^+ \subset R(K,T)$ as in (12). Write $\Pi = \Pi(R^+)$ for the corresponding set of simple roots. Fix also a homomorphism $\phi\colon SU(2) \to K$, and write $\gamma \in X_*(T)$ as in Theorem 7. According to Theorem 6, the $W(K,T)$ orbit $W(K,T)\cdot\gamma \subset X_*(T)$ has a unique dominant element γ'. Such a dominant element determines a collection of non-negative integers

$$\langle \gamma', \alpha \rangle \qquad (\alpha \in \Pi).$$

Conversely, each such collection of non-negative integers determines at most one dominant coweight in the span of the coroots. (They will determine a unique element in the rational span of the coroots; there are some congruence conditions on the integers to get an element in the integer span of the coroots.) Again because of Theorem 7, it follows that this collection of non-negative integers determines ϕ up to conjugacy. The collection of integers (labelling simple roots) is called *the Dynkin diagram of* ϕ. Dynkin's problem was to determine all possible Dynkin diagrams of $SU(2)$ homomorphisms. Here is a key result.

Theorem 8 *Suppose K is a compact connected Lie group, T is a maximal torus in K, and R^+ is a positive root system for $R(K,T)$, with simple roots* Π. *Suppose*

$$\phi\colon SU(2) \to K,$$

is a group homomorphism. Then the corresponding dominant coweight γ' described above attaches to every simple root in Π *the integer 0, 1, or 2. In particular, the total number of K-conjugacy classes of homomorphisms ϕ is at most* $3^{|\Pi|}$.

Proof. We may replace ϕ by a conjugate so that the restriction of ϕ to T_1 is equal to the dominant coweight γ'. Write $d\phi_{\mathbb{C}}$ for the complexified differential of ϕ, a Lie algebra homomorphism from $\mathfrak{su}(2)_{\mathbb{C}}$ to $\mathfrak{k}_{\mathbb{C}}$. We will use the root decompositions of these two algebras from (6) and (8). Fix a basis vector X_α for each root space $\mathfrak{k}_{\mathbb{C}}^\alpha$. Notice first that the action of $\mathrm{Ad}(T_1)$ provides a $\mathbb{Z}$-grading of $\mathfrak{su}(2)_{\mathbb{C}}$:

$$\mathfrak{su}(2)_{\mathbb{C}}[-2] = \mathfrak{su}(2)_{\mathbb{C}}^{-\alpha_1}, \qquad \mathfrak{su}(2)_{\mathbb{C}}[0] = \mathfrak{t}_{1,\mathbb{C}}, \qquad \mathfrak{su}(2)_{\mathbb{C}}[2] = \mathfrak{su}(2)_{\mathbb{C}}^{\alpha_1},$$

and all other levels are zero. Similarly, the action of $\mathrm{Ad}(\phi(T_1)$ provides a $\mathbb{Z}$-grading of $\mathfrak{k}_{\mathbb{C}}$. The zero level is

$$\mathfrak{k}_{\mathbb{C}}[0] = \mathfrak{t}_{\mathbb{C}} \oplus \sum_{\alpha\in R,\ \langle\gamma',\alpha\rangle=0} \mathfrak{k}_{\mathbb{C}}^{\alpha}.$$

If $k < 0$, then

$$\mathfrak{k}_{\mathbb{C}}[k] = \sum_{\alpha\in -R^+,\ \langle\gamma',\alpha\rangle=k} \mathfrak{k}_{\mathbb{C}}^{\alpha},$$

a sum of negative root spaces. If $k > 0$, then

$$\mathfrak{k}_{\mathbb{C}}[k] = \sum_{\alpha\in R^+,\ \langle\gamma',\alpha\rangle=k} \mathfrak{k}_{\mathbb{C}}^{\alpha},$$

a sum of positive root spaces.

Now the Lie algebra homomorphism $d\phi_{\mathbb{C}}$ must respect these gradings. In particular, the element

$$E = d\phi_{\mathbb{C}} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

must belong to $\mathfrak{k}_{\mathbb{C}}[2]$. Consequently E is a sum of positive root vectors:

$$E = \sum_{\alpha\in R^+,\ \langle\gamma',\alpha\rangle=2} e_\alpha X_\alpha. \tag{14}$$

The adjoint action of $SU(2)$ on $\mathfrak{k}_{\mathbb{C}}$ provides a finite-dimensional complex representation of $SU(2)$. Such representations are understood in great detail. We need only one fact: for every $k < 0$, the adjoint action of E provides a one-to-one mapping

$$\mathrm{ad}(E)\colon \mathfrak{k}_{\mathbb{C}}[k] \hookrightarrow \mathfrak{k}_{\mathbb{C}}[k+2]. \tag{15}$$

We can now prove the theorem. Suppose δ is a simple root in Π. We are to show that $\langle\gamma',\delta\rangle \leq 2$. Suppose not; then $\langle\gamma',-\delta\rangle \leq -3$. Therefore the root vector $X_{-\delta} \in \mathfrak{k}_{\mathbb{C}}[k]$, with $k \leq -3$. According to (15), it follows that $[E, X_{-\delta}]$ is a non-zero element of $\mathfrak{k}_{\mathbb{C}}[k+2]$, with $k+2 \leq -1$. Therefore $[E, X_{-\delta}]$ is a non-zero sum of negative root vectors.

On the other hand, (14) says that E is a sum of positive root vectors for roots of level 2 (and therefore *not* including δ). The Lie bracket of a positive root vector and a negative simple root vector is always a positive root vector (except that $[X_\delta, X_{-\delta}] \in \mathfrak{t}_{\mathbb{C}}$). Since this last possiblity has been excluded, we see that $[E, X_{-\delta}]$ is a sum of positive root vectors. This is a contradiction. The conclusion is that $\langle\gamma',\delta\rangle \leq 2$, as we wished to show. Q.E.D.

Theorem 8 shows that the set $D(R^\vee)$ is finite, but does not determine it completely. Dynkin's determination of it is a case-by-case enumeration of the possibilities. His work has been simplified significantly (notably by Bala and Carter), but even today there is no satisfactory *a priori* answer: no characterization of $D(R^\vee)$ by properties analogous to Theorem 8. Here are some candidates for such properties.

Theorem 9 *Suppose K is a compact connected Lie group, T is a maximal torus in K, and R^+ is a positive root system for $R(K,T)$, with simple roots Π. Write $D(R^\vee)$ for the set of Dynkin coweights. Then every dominant element $\gamma \in D(R^\vee)$ has the following properties.*

1. *The coweight γ belongs to the lattice generated by $R^\vee$.*
2. *If δ is a simple root, then $\langle\gamma,\delta\rangle \leq 2$.*
3. *If δ is a simple root such that $\langle\gamma,\delta\rangle = 1$, then there is a root β with $\langle\gamma,\beta\rangle = 2$ such that $\beta - \delta$ is a root.*
4. *The coweight γ is conjugate by $W(K,T)$ to $-\gamma$.*
5. *For every non-negative integer k, the number of roots satisfying $\langle\gamma,\beta\rangle = k$ (plus the dimension of T, if $k = 0$) is greater than or equal to the number of roots satisfying $\langle\gamma,\beta\rangle = k+2$.*
6. *Write R^{even} for the set of roots taking even values on γ, which is a root system in its own right. Then γ satisfies the analogues of 1) and 2) with R replaced by R^{even}.*
7. *Suppose (π, V) is a finite-dimensional representation of the group K. For every non-negative integer k, the number of weights λ of V (counted with multiplicity) satisfying $\langle\gamma,\lambda\rangle = k$ is greater than or equal to the number of weights (with multiplicity) satisfying $\langle\gamma,\lambda\rangle = k+2$.*

I will not prove this in detail, but here are some hints. Of course we already proved 1) and 2) in Theorem 8 and the preceding discussion. Part 3) follows from the proof of 2). Part 4) is a consequence of the fact that the Weyl group of $SU(2)$ acts by inversion on T_1 (together with Theorem 2 to turn conjugacy by $\phi(SU(2))$ into Weyl group conjugacy). Parts 5) and 7) are general properties of representations of $SU(2)$, applied to $\mathrm{Ad} \circ \phi$ and $\pi \circ \phi$ respectively. Part 6) comes by considering the centralizer in K of $\phi(-I)$. This subgroup obviously contains T (since $\phi(-I) \in \phi(T_1) \subset T$) and $\phi(SU(2))$ (since $-I$ is central in $SU(2)$). Its root system is R^{even}.

These conditions provide only negative information: they never say that some homomorphism of $SU(2)$ actually exists. So far our only positive result is Theorem 3, which says that $R^\vee \subset D(R^\vee)$. Here is another.

Theorem 10 *Suppose K is a compact connected Lie group, T is a maximal torus in K, and R^+ is a positive root system for $R(K,T)$, with simple roots Π. Write $\gamma = 2\rho^\vee$ for the sum of all the positive coroots.*

1. *If δ is a simple root, then $\langle \gamma, \delta \rangle = 2$.*
2. *There is a homomorphism $\phi : SU(2) \to K$ whose restriction to T_1 is γ. We have*

$$d\phi_{\mathbb{C}} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \sum_{\alpha \in \Pi} e_\alpha X_\alpha,$$

$$d\phi_{\mathbb{C}} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \sum_{\alpha \in \Pi} e_{-\alpha} X_{-\alpha}.$$

   The constants e_α and $e_{-\alpha}$ are non-zero.

Part 1) is standard (although it is the analogue for roots that is usually considered); a proof may be found in Knapp,[13] Proposition 2.69. Given 1), the construction of the Lie algebra homomorphism $d\phi$ subject to the conditions in 2) is quite easy. We omit the details. The homomorphism ϕ constructed in the theorem is called a *principal* $SU(2)$. A wealth of beautiful properties of it may be found in Kostant.[14]

Theorem 11 *Suppose K is a compact connected Lie group, T is a maximal torus in K, and $R = R(K,T)$ is the root system for T in K. Let S be a subset of R having the properties*

$$S = -S, \qquad (S + S) \cap R \subset S.$$

1. *There is a compact connected subgroup H of K containing T, with root system equal to S.*
2. *The Dynkin set for S is contained in the Dynkin set for R:*

$$D(S^\vee) \subset D(R^\vee) \subset X_*(T).$$

In particular, the sum of any set of positive coroots for S belongs to $D(R^\vee)$.

Part 1) is a straightforward consequence of the relation between the structure constants of the (complexified) Lie algebra $\mathfrak{k}_{\mathbb{C}}$ of K and the root system. Then part 2) is immediate: a homomorphism of $SU(2)$ into H is automatically a homomorphism into K.

Table 1. Possible Dynkin diagrams for A_2

diagram	principal in subsystem	violates Theorem 9
0—0	$\emptyset$	
1—1	A_1	
2—2	A_2	
0—1		1), 3), 4)
0—2		1), 4)
1—0		1), 3), 4)
0—2		1), 4)
1—2		1), 3), 4)
2—0		1), 4)
2—1		1), 3), 4)

Table 2. Possible Dynkin diagrams for C_2

diagram	principal in subsystem	violates Theorem 9
$0 \Leftarrow 0$	$\emptyset$	
$0 \Leftarrow 2$	A_1^{short}, $A_1^{\text{long}} \times A_1^{\text{long}}$	
$1 \Leftarrow 0$	A_1^{long}	
$2 \Leftarrow 2$	C_2	
$0 \Leftarrow 1$		1), 3)
$1 \Leftarrow 1$		1)
$1 \Leftarrow 2$		3)
$2 \Leftarrow 0$		7) (5 diml. rep.)
$2 \Leftarrow 1$		1), 3)

The most obvious subgroups H as in the theorem are the *Levi subgroups* of K. These are the centralizers in K of one-parameter subgroups of T. If $\gamma \in X_*(T)$, then the corresponding subroot system and Levi subgroup are

$$S(\gamma) = \{\alpha \in R \mid \langle \gamma, \alpha \rangle = 0\}, L(\gamma) = K^{\gamma(U(1))}.$$

Such a subroot system always arises as the span of a subset of the simple roots for some positive system in R. But there are other examples as well, like the subsystem of type $C_1 \times C_1$ in C_2.

Let us see what we know about $D(R^\vee)$ in the simplest examples, beginning with $K = SU(2)$. The root system of $SU(2)$ is of type A_1; it has a single positive root α_1, which is simple. We know three elements of $D(R^\vee)$: the

Table 3. Possible Dynkin diagrams for G_2

diagram	principal in subsystem	violates Theorem 9
0⇛0	$\emptyset$	
0⇛1	A_1^{long}	
1⇛0	A_1^{short}	
0⇛2	A_2^{long}	
2⇛2	G_2	
1⇛1		6) ((2) for R^{even})
2⇛0		5) ($k = 4$)
2⇛1		3)
1⇛2		3)

coweight 0 (corresponding to the trivial homomorphism of $SU(2)$ to itself); $\alpha_1^\vee$ (corresponding to the identity homomorhism of $SU(2)$ to itself, which is both principal and attached to a root); and $-\alpha_1^\vee$ (a Weyl group conjugate of $\alpha_1^\vee$; the corresponding homomorphism of $SU(2)$ is inverse transpose). The first two of these are dominant, and give rise to the Dynkin diagrams (0) and (2) respectively (since $\langle \alpha_1^\vee, \alpha_1 \rangle = 2$). The only other Dynkin diagram allowed by Theorem 8 is (1), and the corresponding coweight $\alpha_1^\vee/2$ is not in $X_*(T_1)$. This last possibility is ruled out by Theorem 9(1). So we have determined $D(R^\vee)$ in type A_1: there are three points, corresponding to two Dynkin diagrams.

Next, we turn to $SU(3)$, for which the Dynkin diagram is of type A_2. Table 1 lists the nine Dynkin diagrams allowed by Theorem 9(2). Three of these correspond to homomorphisms of $SU(2)$ given by Theorem 11; the corresponding subroot systems are listed. The remaining six violate some of the necessary conditions in Theorem 9. We list only violations of 1), 3), or 4): the remaining conditions are progressively more difficult to check, and part of the goal of this exercise is seek candidates for a simple and useful characterization of $D(R^\vee)$. (For $SU(n)$, Theorem 9(4) and (7) for the standard n-dimensional representation characterizes $D(R^\vee)$ completely. This is an elementary consequence of the representation theory of $SU(2)$; that is, of the explicit determination of $D(SU(n))$ on a case-by-case basis. It is therefore in the spirit of Dynkin's original work, on which we would like to improve.)

For the group $Sp(4)$, with Dynkin diagram of type C_2, the situation is more difficult. The results are listed in Table 2. Condition (4) of Theorem 9 is automatically satisfied, since -1 is in the Weyl group. The most difficult possible Dynkin diagram is $2 \Leftarrow 0$. The corresponding coweight satisfies Theorem 9 (1)–(6), as well as (7) for the standard 4-dimensional representation. But (7) is not satisfied for the 5-dimensional representation (arising from the covering map $Sp(4) \to SO(5)$), so this coweight does not belong to $D(R^\vee)$.

For the compact group of type G_2, -1 is in the Weyl group, and the coweight lattice is equal to the coroot lattice; so conditions (1) and (4) of Theorem 9 are automatically satisfied. Condition (3) rules out two possible Dynkin diagrams. The remaining two possibilities $1 \Lleftarrow 1$ and $2 \Lleftarrow 0$ are more difficult to rule out, but conditions (5) and (6) suffice.

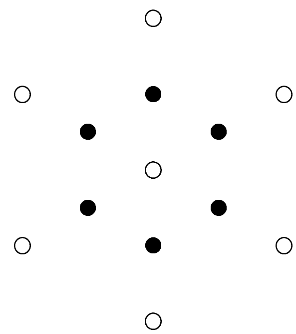
For more complicated groups, Theorems 11 and 10 do not suffice to construct all of $D(R^\vee)$. (They are sufficient in type A and type C. In type B, Theorem 11 constructs every non-principal $SU(2)$ from a proper subgroup, but not necessarily using a principal $SU(2)$ in the subgroup.) Suppose now that $n = p + q$ is written as a sum of two positive integers. Then $SO(n) \supset SO(p) \times SO(q)$, and so the principal $SU(2)$ in $SO(p) \times SO(q)$ becomes an $SU(2)$ in $SO(n)$. If p and q are both odd, this subgroup is not one of those constructed in Theorem 11 (because a maximal torus in $SO(p) \times SO(q)$ is not maximal in $SO(n)$). If p and q are distinct and at least 3, then this $SU(2)$ does not arise from the theorems. (If $p = 1$ and q is odd, then it is the principal $SU(2)$ in $SO(n)$. If $p = q$ is odd, then it is the principal $SU(2)$ in the subgroup $U(p)$ of $SO(2p)$.) The first case of an $SU(2)$ not arising from Theorems 11 and 10 is $n = 8$, $p = 5$, $q = 3$: the principal $SU(2)$ in $SO(5) \times SO(3) \subset SO(8)$. Many more examples appear in the exceptional groups.

We have included illustrations of the Dynkin sets $D(R^\vee)$ for the rank two root systems in Figures 1, 2, and 3. Because of the Weyl group invariance, these figures include a great deal of redundant information; but the symmetry may at the same time make them easier to grasp. In each figure, the points of $D(R^\vee)$ are indicated by circles; the filled circles are the coroots $R^\vee \subset D(R^\vee)$.

4 Split reductive groups

We saw in Theorem 5 that compact connected Lie groups are determined by root data. Arthur's conjecture concerns an analogous family of groups over a local field. We will not try to recall the theory of algebraic groups; but here are a few elementary pieces of it. We begin with a field F (for the moment entirely arbitrary). A *split torus over* F is a group T that is isomorphic to a finite product of copies of the multiplicative group $F^\times$. The number of copies of $F^\times$ is called the *dimension of* T. (The penalty for being so careless about

Figure 1. The set $D(R^\vee)$ for type A_2



the notion of algebraic group is that this dimension is not determined uniquely by T. If for example F is the field with two elements, then any split torus over F is trivial as a group. We will not allow this ambiguity to lead us astray, however.) If X is any lattice of rank n, we can construct an n-dimensional split torus $T(X)$ over F by

$$T(X) = F^\times \otimes_{\mathbb{Z}} X;$$

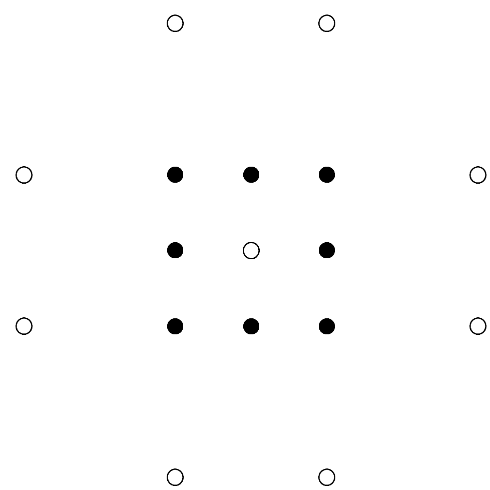
the tensor product is defined using the natural $\mathbb{Z}$-module structure on the abelian group $F^\times$.

Suppose H is a group and A is an abelian group. Recall that a *central extension of H by A* is a group G endowed with a short exact sequence

$$1 \to A \to G \to H \to 1$$

having the property that A is contained in the center of G. A *unipotent algebraic group over F* is a group N obtained from the trivial group by repeated central extensions by F. (This is not the definition of unipotent, but the group of F-points of any unipotent algebraic group over F will have this property.) The number of repetitions of the central extension is called the *dimension of U*. An example of a unipotent group is the group N_n of upper triangular matrices with entries in F and ones on the diagonal. (Let $N_n(k)$ be the subgroup with zeros in the first k rows above the diagonal. Then $N_n(k)$ is a normal subgroup of N_n, and the quotient $N_n(k-1)/N_n(k)$ is isomorphic

Figure 2. The set $D(R^\vee)$ for type C_2



to an $(n-k)$-dimensional vector space over F. Furthermore this subgroup of $N_n/N_n(k)$ is central. It follows that N_n is unipotent, of dimension $n(n-1)/2$.)

Suppose now that $\Psi = (X, R, X^\vee, R^\vee)$ is a reduced root datum, and R^+ is a system of positive roots for Ψ. We will sketch briefly the construction from Ψ of a split reductive algebraic group $G(\Psi, F)$ over F. We first construct the split torus

$$T(X^\vee) = F^\times \otimes_{\mathbb{Z}} X^\vee.$$

Each coweight $\gamma \in X^\vee$ defines in an obvious way a homomorphism (also denoted γ)

$$\gamma \colon F^\times \to T(X^\vee), \qquad \gamma(z) = z \otimes \gamma.$$

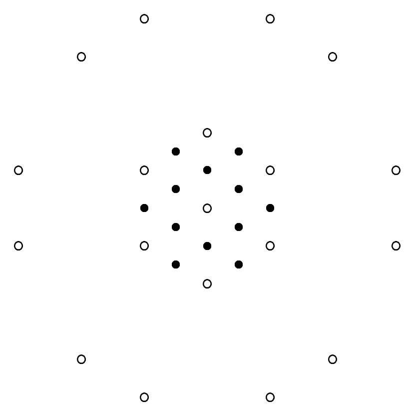
Each weight $\lambda \in X$ defines a character

$$\lambda \colon T(X^\vee) \to F^\times, \qquad \lambda(z \otimes \gamma) = z^{\langle \gamma, \lambda \rangle}. \tag{16}$$

We can define for every $\lambda \in X$ a semidirect product group

$$S_\lambda(F) = T(X^\vee, F)) \ltimes N_\lambda(F).$$

Figure 3. The set $D(R^\vee)$ for type G_2



Set-theoretically this is the product of $T(X^\vee, F)$ and $N_\lambda(F) \simeq F$. The group law is

$$(t, x)(t', x') = (tt', \lambda(t')^{-1}x + x').$$

The group $S_\lambda(F)$ is algebraic; it is the semidirect product of the split torus by the one-dimensional unipotent group $N_\lambda(F)$.

The second step in the construction of $G(\Psi, F)$ is the construction of a Borel subgroup

$$B(\Psi, F) = T(X^\vee, F) \ltimes N(\Psi, F). \tag{17}$$

This is again an algebraic group, a semidirect product of the split torus $T(X^\vee, F)$ with a unipotent group $N(\Psi, F)$ of dimension equal to the cardinality of R^+. In fact $N(\Psi, F)$ is (as a group with $T(X^\vee, F)$ acting as automorphisms) an iterated central extension of the one-dimensional groups $N_\alpha(F)$, for $\alpha \in R^+$. (This does not yet specify the group $N(\Psi, F)$, but we will not describe it more precisely.) In the same way, one can build an opposite Borel subgroup $B^{\mathrm{op}}(\Psi, F) = T(X^\vee, F) \ltimes N^{\mathrm{op}}(\Psi, F)$, with $N^{\mathrm{op}}(F)$ built from the negative roots.

Finally, the group $G(\Psi, F)$ may be taken as the free group generated by $B(\Psi, F)$ and $B^{\mathrm{op}}(\Psi, F)$, subject to some simple relations: for example, the identification of the two copies of $T(X^\vee, F)$, and other relations satisfied by the subgroups of upper and lower triangular matrices in $SL(2, F)$. For more details of the construction of $G(\Psi, F)$, see for example Springer.[20]

Theorem 12 *Suppose $\Psi = (X, R, X^\vee, R^\vee)$ is a reduced root datum, R^+ is a set of positive roots for R, and F is a field. Then there is a reductive algebraic group $G(\Psi, F)$ of dimension equal to the cardinality of R plus the rank of X. This group is generated by two solvable algebraic subgroups $B(\Psi, F) = T(X^\vee, F) \ltimes N(\Psi, F)$ and $B^{\mathrm{op}}(\Psi, F) = T(X^\vee, F) \ltimes N^{\mathrm{op}}(\Psi, F)$ described above. The subset $N(\Psi, F)T(X^\vee, F)N^{\mathrm{op}}(\Psi, F)$ is identified by the group multiplication with $F^{|R^+|}(F^\times)^{\mathrm{rank}(X)}F^{|R^+|}$; it is a Zariski dense open subset of $G(\Psi, F)$.*

If F is a local field, then the locally compact topology on F defines by the identification above a locally compact topology on $G(\Psi, F)$. In this topology, the quotient space $G(\Psi, F)/B(\Psi, F)$ is compact.

From now on we fix a local field F, and a split reductive group $G(\Psi, F)$ as in Theorem 12. We wish to describe a certain family of representations of $G(\Psi, F)$. The simplest reductive group (attached to the root datum $(\mathbb{Z}, \emptyset, \mathbb{Z}, \emptyset)$) is the multiplicative group $F^\times$. Because it is abelian, its irreducible representations are one-dimensional; they are just the homomorphisms of $F^\times$ into $\mathbb{C}^\times$, also called *quasicharacters*. Quasicharacters can be described completely and explicitly, but we will need only certain special ones. The first is the *absolute value on F*,

$$|\cdot|: F^\times \to \mathbb{R}^\times. \tag{18}$$

By definition $|z|$ is the scalar by which multiplication by z changes an additive Haar measure on F. It is the usual absolute value on $\mathbb{R}^\times$, and the square of the usual absolute value on $\mathbb{C}^\times$. For a p-adic field its values are just the powers of q, the order of the residue field. More details may be found in Weil.[26]

Since the absolute value takes positive real values, we may form arbitrary real exponentials (or even complex exponentials) of it. In this way we can get a real line of quasicharacters of $F^\times$:

$$|\cdot|^s: F^\times \to \mathbb{R}^\times.$$

Next we consider quasicharacters of the torus $T(X^\vee, F) = F^\times \otimes_{\mathbb{Z}} X^\vee$. We saw already in (16) that every element $\lambda \in X$ can be regarded as a homomorphism from $T(X^\vee, F)$ to $F^\times$. Composing with the absolute value map gives quasicharacters

$$|\lambda|: T(X^\vee, F) \to \mathbb{R}^\times.$$

Because of the possibility of forming real exponentials, we can do even better, however. The *real dual space for the torus* $T(X^\vee, F)$ is by definition the real vector space

$$\mathfrak{t}(X) = \mathbb{R} \otimes_{\mathbb{Z}} X. \tag{19}$$

This is a real vector space of dimension equal to the rank of X (which is the dimension of $T(X^\vee, F)$). A typical element is a finite sum $\nu = \sum s_i \otimes \lambda_i$, with s_i real and λ_i in X. To every such element ν we can associate a quasicharacter (denoted by the same letter)

$$\nu: T(X^\vee, F) \to \mathbb{R}^\times, \qquad \nu(t) = \prod_i |\lambda_i(t)|^{s_i}. \tag{20}$$

Theorem 13 *Suppose X and $X^\vee$ are a lattice and the dual lattice, and F is a local field. Write $T(X^\vee, F)$ for the corresponding split torus, and $\mathfrak{t}(X)$ for the real dual space defined in (19). Then the quasicharacters ν defined above (for $\nu \in \mathfrak{t}(X)$) are well-defined and distinct. They take positive real values, and in fact exhaust the quasicharacters with this property.*

Ultimately we are interested in unitary representations. In the case of tori, the quasicharacter ν is unitary only for $\nu = 0$ (when it is trivial). An interesting connection with unitary representations will appear only when we pass to the reductive group $G(\Psi, F)$. This we now do. Because of the semidirect product decomposition (17), any quasicharacter of $T(X^\vee, F)$ extends immediately to a quasicharacter of $B(\Psi, F)$, with $N(\Psi, F)$ acting trivially.

We are now in a position to apply Mackey's induction construction to get a representation of $G(\Psi, F)$. Phrased geometrically, the idea is this. (The term *manifold* in the following discussion means a space locally homeomorphic to F^n. Much of the basic theory of real manifolds carries over to manifolds over any local field F.) The quasicharacter ν of $B(\Psi, F)$ defines an equivariant line bundle $\mathcal{L}(\nu)$ over the homogeneous space $G(\Psi, F)/B(\Psi, F)$. (The fiber at the base point $eB(\Psi, F)$ is $\mathbb{C}$, and the action of the isotropy group $B(\Psi, F)$ on this fiber is by the quasicharacter ν. We can then define

$$\mathcal{H}(\Psi, \nu) = \mathrm{Ind}_{B(\Psi,F)}^{G(\Psi,F)}(\nu) = \text{space of sections of } \mathcal{L}(\nu + \rho) \tag{21}$$

The action of $G(\Psi, F)$ on this space is by left translation; it is written $\pi(\Psi, \nu)$, and called a *real unramified principal series representation.*

Two points require explanation. First, the shift ρ is equal to half the sum of the positive roots:

$$\rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha \in \mathfrak{t}(X).$$

Because of this shift, and more importantly because of the appearance of $B(\Psi, F)$, the representation $\mathcal{H}(\Psi, \nu)$ depends not just on the root datum but also on the choice of positive roots. Second, we need to say exactly which sections of $\mathcal{L}(\nu + \rho)$ constitute the representation space. This is a little subtle. If F is p-adic, a reasonable choice is the *locally constant* sections: those invariant under left translation by some compact open subgroup of $G(\Psi, F)$ (depending on the section). If F is archimedean, one can consider the real analytic sections. Different choices are appropriate for different purposes, and we will be somewhat vague about this point.

The line bundle $\mathcal{L}(\rho)$ may be identified with the bundle of half densities on the manifold $G(\Psi, F)/B(\Psi, F)$. (This is a fairly elementary consequence of the structure theory in Theorem 12.) In particular, there is a natural and $G(\Psi, F)$-invariant pre-Hilbert space structure on $\mathcal{H}(\Psi, 0)$. (To form the inner product of two sections, one essentially takes the tensor product of the first with the complex conjugate of the second. This tensor product is a section of the density bundle; that is, it is a nice measure on the compact manifold $G(\Psi, F)/B(\Psi, F)$. The inner product is the total mass of this measure.) In this way $\pi(\Psi, 0)$ may be regarded as a unitary representation. This is just a very special case of Mackey's unitary induction construction, building a unitary representation of $G(\Psi, F)$ from the unitary character 0 of the subgroup $B(\Psi, F)$.

There is another unitary representation hiding here. If $\nu = -\rho$, then $\mathcal{H}(\Psi, -\rho)$ is by definition the space of sections of the trivial line bundle $\mathcal{L}(-\rho+\rho)$; that is, the space of functions on $G(\Psi, F)/B(\Psi, F)$. This space contains the one-dimensional space of constant functions as a $G(\Psi, F)$-invariant subspace; so $\pi(\Psi, -\rho)$ contains the (unitary) trivial representation of $G(\Psi, F)$ as a subrepresentation.

We can get the same representation in a different way. If $\nu = \rho$, then $\mathcal{H}(\Psi, \rho)$ is by definition the space of sections of the line bundle $\mathcal{L}(\rho+\rho)$, which is the space of densities on $G(\Psi, F)/B(\Psi, F)$. There is a $G(\Psi, F)$-equivariant map from this space to the complex numbers (with trivial $G(\Psi, F)$ action), sending each density to its total mass. That is, $\pi(\Psi, \rho)$ contains the trivial representation of $G(\Psi, F)$ as a quotient.

Here are some general facts about the representations $\pi(\Psi, \nu)$.

Theorem 14 *Suppose* $\Psi = (X, R, X^\vee, R^\vee)$ *is a reduced root datum,* R^+ *is a set of positive roots for* R*, and* F *is a local field. For each* $\nu \in \mathfrak{t}(X)$ *(see (19)), define a principal series representation* $(\pi(\Psi, \nu), \mathcal{H}(\Psi, \nu)$ *of* $G(\Psi, F)$ *as in (21).*

1. *The representation* $\pi(\Psi, \nu)$ *has a finite composition series. There is a*

distinguished irreducible composition factor $(\overline{\pi}(\Psi,\nu), \overline{\mathcal{H}}(\Psi,\nu))$*, called the* Langlands subquotient.

2. *Suppose ν and ν' are in $\mathfrak{t}(X)$. Then the following conditions are equivalent.*
   - *There is a $w \in W$ (the Weyl group of the root system R) so that $w\nu = \nu'$.*
   - *The representations $\pi(\Psi,\nu)$ and $\pi(\Psi,\nu')$ have isomorphic irreducible composition factors (appearing with the same multiplicities).*
   - *The irreducible representations $\overline{\pi}(\Psi,\nu)$ and $\overline{\pi}(\Psi,\nu')$ are equivalent.*

The history of this theorem is long, complicated, and occasionally bitter; I will not try to sort it out here. The distinguished composition factor $\overline{\pi}(\Psi,\nu)$ can be characterized in at least two completely different ways. One is analytic: it is the composition factor whose matrix coefficients have the largest growth at infinity on $G(\Psi,F)$.

The second characterization is algebraic, and takes a bit longer to state. There is a natural compact subgroup $K(\Psi,F)$ of $G(\Psi,F)$. If F is p-adic, this subgroup is the "integer points" of the algebraic group $G(\Psi,F)$. (For example, in the case of $GL(n)$, it is the group of $n\times n$ matrices with coefficients in the ring of integers of F, such that the determinant is an integer of norm 1.) If F is archimedean, $K(\Psi,F)$ is a maximal compact subgroup of $G(\Psi,F)$. (For example, in the case of $GL(n,\mathbb{R})$, it is the group of orthogonal matrices.) The restriction of $\pi(\Psi,\nu)$ to $K(\Psi,F)$ contains the trivial representation of $K(\Psi,F)$ exactly once. (This is an easy consequence of the standard decomposition $G(\Psi,F) = K(\Psi,F)B(\Psi,F)$.) It follows that $\pi(\Psi,\nu)$ contains a unique irreducible composition factor having a $K(\Psi,F)$-fixed vector. This composition factor is $\overline{\pi}(\Psi,\nu)$.

Theorem 14 allows us to regard the real vector space $\mathfrak{t}(X)$ as parametrizing a set of irreducible representations $\{\overline{\pi}(\Psi,\nu)\}$ in a W-invariant way. We have seen that at least some of these representations are unitary—more precisely, that they admit $G(\Psi,F)$-invariant pre-Hilbert space structures that can be completed to provide unitary representations. Because of the central role of unitary representations in problems of harmonic analysis, it is natural to define

$$P(\Psi,F) = \{\nu \in \mathfrak{t}(X) | \overline{\pi}(\Psi,\nu) \text{has } G(\Psi,F)\text{-invt pre-Hilbert structure}\}. \quad (22)$$

For each such element ν, the representation $\overline{\pi}(\Psi,\nu)$ can be completed to a unitary representation of $G(\Psi,F)$. In light of Theorem 14, it follows that

$P(\Psi, F)/W$ may be regarded as a subset of the unitary dual of $G(\Psi, F)$. Those few examples that are understood indicate that this subset is analytically the most difficult part of the unitary dual to understand. (There are also formidable algebraic difficulties in understanding the unitary dual, related to the arithmetic of the local field F.) The "classical representation-theoretic problem" mentioned in the introduction is

$$\textbf{Spherical unitary dual problem:} \quad \text{calculate } P(\Psi, F). \tag{23}$$

Here are some representative cases in which $P(\Psi, F)$ is known. For Ψ of type A, it was completely determined by Tadić[23] in the case of p-adic F and by Vogan[24] for archimedean F. (All of the unitary representations involved in these cases had essentially been constructed by Stein;[21] what was missing was a proof that there were no more.) For Ψ classical and F complex, $P(\Psi, F)$ was determined by Barbasch.[2] For Ψ classical and F p-adic, $P(\Psi, F)$ was determined by Barbasch and Moy.[4]. For Ψ of rank 2 and F complex, $P(\Psi, F)$ was determined by Duflo.[9] In type G_2, the real case was treated by Vogan[25] and the p-adic case by Muić.[17]

Here are some general properties of the set $P(\Psi, F)$.

Theorem 15 *Suppose $\Psi = (X, R, X^\vee, R^\vee)$ is a reduced root datum, R^+ is a set of positive roots for R, and F is a local field. Write $\mathfrak{t}(X)$ for the real dual space of $T(X^\vee, F)$ defined in (19), and W for the Weyl group of R. Finally, write $\rho \in \mathfrak{t}(X)$ for half the sum of the roots in R^+.*

1. *The set $P(\Psi, F)$ is a compact W-invariant polyhedron contained in the real span of the roots R. This polyhedron depends only on R and F (and not on the lattices X and $X^\vee$).*

2. *Suppose ν is in $P(\Psi, F)$. Then ν is conjugate by W to $-\nu$.*

3. *The polyhedron $P(\Psi, F)$ is contained in the convex hull of $W \cdot \rho$.*

4. *The element ρ belongs to $P(\Psi, F)$; the corresponding unitary representation of $G(\Psi, F)$ is the trivial representation.*

5. *Suppose $\Psi_L = (X, R_L, X^\vee, R_L^\vee)$ is a Levi subsystem of Ψ. (This means that R_L consists of those roots vanishing on a fixed coweight $\gamma \in X^\vee$). Then $P(\Psi_L, F) \subset P(\Psi, F)$. In particular, $P(\Psi, F)$ contains the half sum of the positive roots R_L^+.*

Again this is an old result, difficult to attribute precisely. Here are some of the ideas needed for the proofs. Part (2) is the easiest: the condition

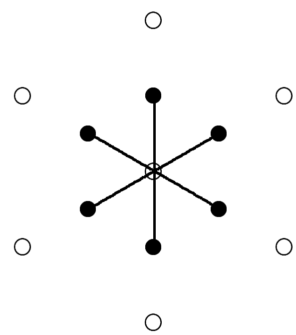
$-\nu \in W \cdot \nu$ is necessary and sufficient for $\overline{\pi}(\Psi, \nu)$ to admit a $G(\Psi, F)$-invariant Hermitian form. (Using integration of densities over $G(\Psi, F)/B(\Psi, F)$, it is easy to identify the Hermitian dual of $\pi(\Psi, \nu)$ as $\pi(\Psi, -\nu)$. It follows (from the algebraic characterization of the Langlands subquotient) that the Hermitian dual of $\overline{\pi}(\Psi, \nu)$ is $\overline{\pi}(\Psi, -\nu)$. Now use Theorem 14.) Part (3) is essentially due to Helgason and Johnson;[11] their proof in the case of archimedean F carries over with little change to the p-adic case. (What they show is that ν belonging to the convex hull of $W \cdot \rho$ is necessary and sufficient for the spherical function—the matrix coefficient given by the $K(\Psi, F)$-fixed vector—to be bounded.) Part (4) was explained before Theorem 14.

Part (5) is based on Mackey's idea of "induction by stages." Because of the W-invariance of $P(\Psi, F)$, we may as well assume that the coweight $\gamma \in X^\vee$ defining Ψ_L is dominant for R^+. In that case there is a parabolic subgroup $P = LU$ containing $B(\Psi, F)$, with the property that L is isomorphic to the reductive group $G(\Psi_L, F)$. In this situation the representation $\pi(\Psi, \nu)$ of $G(\Psi, F)$ may be obtained from $\pi(\Psi_L, \nu)$ on L by a trivial extension to U, then unitary induction. It follows that $\overline{\pi}(\Psi, \nu)$ is a subquotient of $\mathrm{Ind}_P^{G(\Psi,F)} \overline{\pi}(\Psi_L, \nu)$. If $\overline{\pi}(\Psi_L, \nu)$ is unitary—that is, if $\nu \in P(\Psi_L, F)$—then $\overline{\pi}(\Psi, \nu)$ must be unitary as well.

Part (1) is the most difficult result here. That $P(\Psi, F)$ depends only on R and F is fairly easy, using the covering maps between the various groups arising. The rest of the statement is based on the construction of the invariant Hermitian form on $\pi(\Psi, \nu)$ using the integral intertwining operators introduced by Schiffmann. Important ideas came from Speh, and the application to unitary representations was mostly developed by Knapp and Stein. What follows from this construction is that $P(\Psi, F)$ is a closed polyhedron in $\{\nu \in \mathfrak{t}(X) \mid -\nu \in W \cdot \nu\}$. The faces of this polyhedron are built from certain hyperplanes of the form $\{\nu \mid \langle \alpha^\vee, \nu \rangle = m\}$. Here $\alpha^\vee \in R^\vee$ is a coroot and m is a non-zero integer. (If F is p-adic, m must be one; if F is real, m must be odd; and if F is complex, m can be any non-zero integer.) The compactness follows from part (3). For more details, the reader may consult Chapter 16 of Knapp.[12]

Because of Theorem 15, we may write $P(R, F)$ instead of $P(\Psi, F)$. Illustrations of the unitary sets $P(R, F)$ for the rank two root systems may be found in Figures 4, 5, and 6. We have omitted the case $F = \mathbb{C}$ for types A_2 and C_2; they may be found in Duflo.[9] (They contain some additional points and intervals.) Just as for $D(R^\vee)$, these figures include a great deal of redundant information; but the symmetry may again make them easier to grasp. In each figure, the points of $P(R, F)$ are indicated by circles, heavy lines and hatched regions. One half of each root (which belongs to $P(R, F)$ by Theorem 15(5))

Figure 4. The set $P(R)$ for type A_2



is indicated by a filled circle. We will explain the open circles in a moment.

A comparison of the figures for $P(R, F)$ and $D(R)$ suggests (for any reduced root system R and any local field F)

$$\textbf{Arthur's conjecture:} \qquad D(R) \subset 2P(R, F). \tag{24}$$

This containment (still unproven in general) is an important special case of conjectures of Arthur.[1] The circles in the figures for $P(R, F)$ are the points of $(1/2)D(R)$; they are contained in $P(R, F)$. (The conjecture (24) is true in all the cases mentioned before Theorem 15, in which $P(R, F)$ has been computed. For p-adic F, it is Theorem 8.1 of Barbasch and Moy.[3])

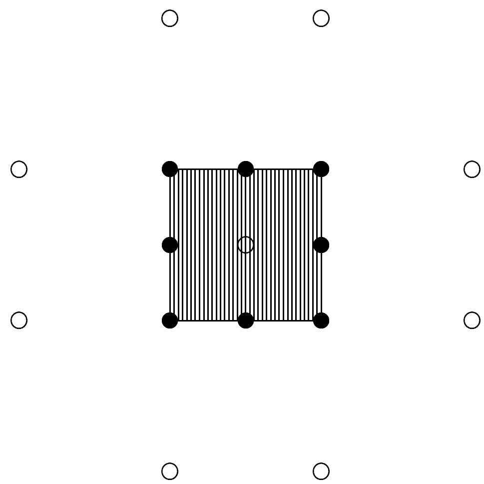
The figures here even support a stronger conjecture

$$D(R) = 2P(R, F) \cap \mathbb{Z}R$$

But this stronger conjecture is false in type C_2 for F complex. (The even part of the metaplectic representation of $G(\Psi, \mathbb{C})$ is isomorphic to a certain $\pi(\Psi, \nu)$, with $2\nu \in \mathbb{Z}R$ but $2\nu \notin D(R)$. For other local fields, the metaplectic representation is not isomorphic to any $\pi(\Psi, \nu)$.)

Arthur's motivation for his conjectures came from the Langlands philosophy of automorphic representations, and from Arthur's trace formula. Roughly speaking, he suggested that the unitary representations whose existence is implied by (24) ought to arise as local components in the residual

Figure 5. The set $P(R, F)$ for type C_2 (F real or p-adic)

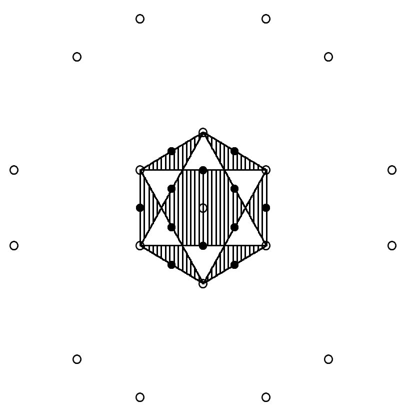


spectrum for a homogeneous space $G(\Psi, \mathbb{A})/G(\Psi, k)$. (Here k is a global field and $\mathbb{A}$ is the corresponding adele ring.)

Let us look carefully at what (24) says. Obviously it concerns a reduced root system R. The right side concerns unitary representations of a split algebraic reductive group $G(\Psi, F)$ with root system R. The left side concerns homomorphisms of $SU(2)$ into a compact Lie group with *coroot system* R. The most basic example says that homomorphisms of $SU(2)$ into $U(n)$ should give rise to unitary representations of $GL(n, F)$ (with n any local field). Another classical example says that homomorphisms of $SU(2)$ into an odd orthogonal group $SO(2n+1)$ should give rise to unitary representations of a symplectic group $Sp(2n, F)$. In every case, the most complicated homomorphism of $SU(2)$ (the principal $SU(2)$, described by Theorem 10) corresponds to the trivial representation of $G(\Psi, F)$.

Without proving Arthur's conjecture, we can ask whether the tools described in Section 3 for understanding $D(K)$ have analogues that help to understand the set of unitary representations $P(R, F)$. Our first step for Dynkin's problem was to identify $D(K)$ with a set of Weyl group orbits on

Figure 6. The set $P(R,F)$ for type G_2 (F real or p-adic)



the lattice $X_*(T)$ (and eventually inside the coroot lattice $\mathbb{Z}R$). (Recall that the coroots of K are the roots of $G(\Psi, F)$.) Correspondingly, Theorem 14 identifies a certain set of unitary representations with a set of Weyl group orbits on the real vector space spanned by R. Next, we can look for analogues of the constraints in Theorem 9 in the setting of unitary representations. Part (1) of Theorem 9 is analogous to the (much weaker) assertion (1) in Theorem 15. Part (4) of Theorem 9 is identical to Theorem 15(2). Part (2) of Theorem 9 is a much stronger version of Theorem 15(3). Parts (3) and (5)–(7) of Theorem 9 have no analogue in Theorem 15.

For the construction of elements of $P(R,F)$, the situation is even worse. Theorem 15(4) does provide an analogue of Theorem 10. But the "functoriality theorem" Theorem 11 is far stronger than Theorem 15(5); the difference is that in the former we can use very general root subsystems of R, but in the latter we must use only Levi subsystems.

We can emerge from this exercise with a goal: to establish more analogues for $P(R,F)$ of results about the Dynkin set $D(R)$. Here is an example.

Theorem 16 *(Barbasch, Vogan) Suppose* $\Psi = (X, R, X^\vee, R^\vee)$ *is a reduced root datum,* R^+ *is a set of positive roots for* R*, and* F *is a local field. Suppose* $\nu \in P(R,F)$ *is dominant; that is, that* $\langle \alpha^\vee, \nu \rangle \geq 0$ *for all* $\alpha \in R^+$*. Then for every simple root* δ *for* R^+*, we have* $\langle \delta^\vee, \nu \rangle \leq 1$.

This is a good analogue of Theorem 9(2). We sketch a proof in Sec. 5.

5 The results of Barbasch and Moy

We begin this section with a summary of some results of Barbasch and Moy[3] in the p-adic case. These results make the calculation of $P(R, F)$ into a finite algebraic problem for each root system R, and show incidentally that $P(R, F)$ is *independent* of the p-adic field F.

We fix therefore a p-adic field F, and write A for the ring of integers in F and $\mathfrak{m} \subset A$ for the maximal ideal of A. The quotient field

$$\overline{F} = A/\mathfrak{m} \tag{25}$$

is a finite field. As mentioned after Theorem 14, the construction of the reductive group $G(\Psi, F)$ gives rise naturally to a compact open subgroup loosely described as

$$K(\Psi, F) = \text{points of } G(\Psi, F) \text{ with coefficients in } A.$$

Barbasch and Moy study the restrictions to $K(\Psi, F)$ of the representations $\pi(\Psi, \nu)$. The quotient map (25) defines a surjective group homomorphism

$$K(\Psi, F) \to G(\Psi, \overline{F}). \tag{26}$$

The group on the right is a finite Chevalley group. We write $K_1(\Psi, F)$ for the kernel of this homomorphism. The finite Chevalley group has its own Borel subgroup $B(\Psi, \overline{F})$, and there is also a short exact sequence

$$1 \to K_1(\Psi, F) \cap B(\Psi, F) \to K(\Psi, F) \cap B(\Psi, F) \to B(\Psi, \overline{F}) \to 1$$

A central role is played by the *Iwahori subgroup*

$$I(\Psi, F) = \text{inverse image of } B(\Psi, \overline{F}); \tag{27}$$

this is the subgroup of $K(\Psi, F)$ generated by $K(\Psi, F) \cap B(\Psi, F)$ and $K_1(\Psi, F)$.

We have already mentioned the decomposition

$$G(\Psi, F) = K(\Psi, F)B(\Psi, F).$$

As a consequence of this decomposition and the definition (21), we find

$$\mathcal{H}(\Psi, \nu)|_{K(\Psi,F)} = \mathrm{Ind}_{K(\Psi,F)\cap B(\Psi,F)}^{K(\Psi,F)}(\nu|_{K(\Psi,F)\cap B(\Psi,F)}). \tag{28}$$

Because the quasicharacter ν takes positive real values, it must be trivial on the compact group $K(\Psi, F) \cap B(\Psi, F)$. Therefore

$$\mathcal{H}(\Psi, \nu)|_{K(\Psi,F)} = \mathrm{Ind}_{K(\Psi,F)\cap B(\Psi,F)}^{K(\Psi,F)}(1),$$

the space of functions on $K(\Psi,F)/(K(\Psi,F)\cap B(\Psi,F))$. We will look at the subspace $\mathcal{H}(\Psi,\nu)^1$ of functions invariant under the normal subgroup $K_1(\Psi,F)$. Clearly this is the same as functions invariant on the right by $K_1(\Psi,F)(K(\Psi,F)\cap B(\Psi,F))$; that is, functions on the homogeneous space $K(\Psi,F)/I(\Psi,F)$. Formally,

$$\mathcal{H}(\Psi,\nu)^1 = \mathrm{Ind}_{I(\Psi,F)}^{K(\Psi,F)}(1). \tag{29}$$

We now analyze $\mathcal{H}(\Psi,\nu)^1$ as a representation of $K(\Psi,F)$. By definition the normal subgroup $K_1(\Psi,F)$ acts trivially, so it is natural to regard the space as a representation of the finite Chevalley group $G(\Psi,\overline{F})$. From this point of view, we easily find

$$\mathcal{H}(\Psi,\nu)^1 = \mathrm{Ind}_{B(\Psi,\overline{F})}^{G(\Psi,\overline{F})}(1).$$

Theorem 17 *Suppose $\Psi = (X,R,X^\vee,R^\vee)$ is a reduced root datum, R^+ is a set of positive roots for R, and $\overline{F}$ is a finite field. Then the irreducible consitutuents of the representation*

$$\mathrm{Ind}_{B(\Psi,\overline{F})}^{G(\Psi,\overline{F})}(1)$$

are naturally parametrized by the irreducible representations of the Weyl group W of the root system. The representation $\sigma(\tau)$ corresponding to a W representation τ appears with multiplicity equal to the dimension of τ.

This theorem is due to Tits. A complete account appears in the unpublished lecture notes of Steinberg;[22] one can also find information in Curtis, Iwahori, and Kilmoyer,[7] or in Chapter 10 of Carter.[5] In the bijection of the theorem, $\sigma(1)$ is the trivial representation of $G(\Psi,\overline{F})$, corresponding to the trivial representation of $K(\Psi,F)$. We had already used the fact that it appears exactly once in $\pi(\Psi,F)$ in giving an algebraic characterization of the Langlands subquotient after Theorem 14.

We can state Theorem 17 as

$$\mathcal{H}(\Psi,\nu)^1 \simeq \sum_{\tau\in\widehat{W}} \dim(\tau)\sigma(\tau). \tag{30}$$

This describes the space of $K_1(\Psi,F)$ invariants in a principal series representation, as a representation of $K(\Psi,F)$. The Langlands subquotient $\overline{\pi}(\Psi,\nu)$ defines a subquotient

$$\overline{\mathcal{H}}(\Psi,\nu)^1 \simeq \sum_{\tau\in\widehat{W}} m(\tau,\nu)\sigma(\tau).$$

The multiplicity $m(\tau, \nu)$ is between 0 and $\dim(\tau)$; it is 1 for τ trivial. If the Langlands subquotient admits an invariant Hermitian form (that is, if $-\nu \in W \cdot \nu$), then this form has a certain signature $(p(\tau, \nu), q(\tau, \nu))$ on each representation $\sigma(\tau)$. The positive and negative parts are non-negative integers, and $p(\tau, \nu)+q(\tau, \nu) = m(\tau, \nu)$. We may normalize the form to be positive on the spherical vector: that is, $p(1, \nu) = 1$, $q(1, \nu) = 0$. The main theorem of Barbasch and Moy[3] is

Theorem 18 *Suppose $\Psi = (X, R, X^\vee, R^\vee)$ is a reduced root datum, R^+ is a set of positive roots for R, and F is a p-adic field. Fix $\nu \in \mathfrak{t}(X)$, and assume that $-\nu \in W \cdot \nu$ (so that $\overline{\pi}(\Psi, \nu)$ admits an invariant Hermitian form). Then the form is positive—equivalently, $\nu \in P(R, F)$—if and only if it is positive on the subspace $\overline{\mathcal{H}}(\Psi, \nu)^1$ of $K_1(\Psi, F)$-fixed vectors. This positivity on the subspace in turn is equivalent to the vanishing of the integers $q(\tau, \nu)$ defined above for every representation τ of W.*

Of course the "only if" part of the theorem is obvious, and the last equivalence is just the definition of $q(\tau, \nu)$. What requires proof is that any negativity of the Hermitian form can be detected inside $\overline{\mathcal{H}}(\Psi, \nu)^1$. This Barbasch and Moy prove by a deep analysis of the reducibility of these principal series representations, using the geometric ideas of Kazhdan and Lusztig.

Theorem 18 reduces the determination of whether ν belongs to $P(\Psi, F)$ to a calculation of the signatures of a family of Hermitian forms, one for each representation of W. Next we will explain how these forms can actually be calculated. We therefore fix a dominant element

$$\nu \in \mathfrak{t}(X), \qquad -\nu \in W \cdot \nu.$$

Let w_0 be the long element of the Weyl group, characterized by $w_0(R^+) = -R^+$. Because ν is dominant, we have $w_0 \cdot \nu = -\nu$. We know that W is generated by reflections in simple roots. Choose an expression of minimal length

$$w_0 = s_N \cdots s_1,$$

with s_i the reflection in some simple root α_i. Define

$$x_r = s_{r-1} s_{r-2} \cdots s_1 \in W \qquad (1 \le r \le N)$$

$$\nu_r = \langle \nu, (x_r \alpha_r)^\vee \rangle \in \mathbb{R}.$$

It is not very difficult to show that

$$\{x_r \alpha_r \mid 1 \le r \le N\} = R^+$$

(with no repetitions). In particular, all of the real numbers ν_r are non-negative. Finally, define

$$A(\nu) = (1 + \nu_N s_N)(1 + \nu_{N-1} s_{N-1}) \cdots (1 + \nu_1 s_1) \in \mathbb{R}W. \tag{31}$$

This element of the group algebra turns out to be independent of the choice of reduced decomposition of w_0, and to be "self-adjoint" (in the sense that the coefficient of each group element w is equal to the complex conjugate of the coefficient of w^{-1}). Consequently $A(\nu)$ defines a self-adjoint operator $\tau(A(\nu))$ on each irreducible representation τ of W.

Theorem 19 *Suppose $\Psi = (X, R, X^\vee, R^\vee)$ is a reduced root datum, R^+ is a set of positive roots for R, and F is a p-adic field. Fix a dominant weight $\nu \in \mathfrak{t}(X)$, and assume that $-\nu \in W \cdot \nu$ (so that $\overline{\pi}(\Psi, \nu)$ admits an invariant Hermitian form.*

1. *The multiplicity $m(\tau, \nu)$ of the representation $\sigma(\tau)$ of $K(\Psi, F)$ in $\overline{\pi}(\Psi, \nu)$ is equal to the rank of the operator $\tau(A(\nu))$.*
2. *The positive part $p(\tau, \nu)$ of the signature on $\sigma(\tau)$ is equal to the number of strictly positive eigenvalues of $\tau(A(\nu))$.*
3. *The negative part $q(\tau, \nu)$ of the signature on $\sigma(\tau)$ is equal to the number of strictly negative eigenvalues of $\tau(A(\nu))$.*

In particular, the Hermitian form is positive—equivalently, $\nu \in P(R, F)$—if and only if $A(\nu)$ has non-negative eigenvalues on the group algebra $\mathbb{C}W$.

This result (due to Barbasch and Vogan) is a fairly easy consequence of the Knapp-Stein construction of the Hermitian form using integral intertwining operators, together with the standard factorization of such operators and a calculation in $SL(2)$. One consequence (which was already evident from the paper of Barbasch and Moy[3]) is that $P(R, F)$ is independent of the p-adic field F.

Here is an example of the operator $A(\nu)$. Suppose Ψ is of type A_2, so that G is locally $SL(3)$. There are two simple reflections s and s' in W; a reduced expression for w_0 is $ss's$. The only dominant elements ν with $-\nu \in W \cdot \nu$ are the multiples of ρ; so we put $\nu = t\rho$, with $t \geq 0$. Then it is not hard to calculate

$$A(t\rho) = (1 + t^2) + 2t(s + s') + 2t^2(ss' + s's) + 2t^3(ss's)$$

To determine $P(R, F)$, we need to know for which non-negative t this operator is non-negative on every representation of W. On the trivial representation of W, it acts by the scalar

$$1 + 4t + 5t^2 + 2t^3 = (1+t)^2(1+2t),$$

which is positive for all $t \geq 0$. On the sign representation, it acts by the scalar

$$1 - 4t + 5t^2 - 2t^3 = (1-t)^2(1-2t).$$

This is non-negative for $0 \leq t \leq 1/2$ and for $t = 1$. We leave to the reader the task of writing down the 2×2 matrix by which $A(t\rho)$ acts in the reflection representation, and verifying that it is non-negative exactly for $0 \leq t \leq 1/2$ and $t = 1$. (As a hint, we note that the eigenvalues of this matrix are

$$(1+t)(1-t)(1+2t), \quad (1+t)(1-t)(1-2t).)$$

The simplicity of all of these formulas for the eigenvalues suggests that there might be comprehensible closed formulas for all of the eigenvalues of $\tau(A(\nu))$ in general. Such formulas would be enormously useful in the study of unitary representations, because of Theorems 19 and 20.

We conclude this paper with an application of the idea of Barbasch and Moy to archimedean fields. For simplicity we will discuss only the complex case; the real case is very similar. We are therefore considering a complex reductive algebraic group $G(\Psi, \mathbb{C})$. The maximal compact subgroup $K(\Psi, \mathbb{C})$ is just the compact connected Lie group associated to Ψ by Theorem 5. This group meets the Borel subgroup in a compact maximal torus:

$$K(\Psi, \mathbb{C}) \cap B(\Psi, \mathbb{C}) = T$$

Just as in the p-adic case, we deduce

$$\mathcal{H}(\Psi, \nu)|_{K(\Psi,\mathbb{C})} = \mathrm{Ind}_T^{K(\Psi,\mathbb{C})}(1). \tag{32}$$

Consequently the multiplicity in $\mathcal{H}(\Psi, \nu)$ of any irreducible representation σ of $K(\Psi, \nu)$ is equal to the dimension of the "zero weight space" σ^T.

Now the normalizer of T in $K(\Psi, \mathbb{C})$ acts on σ^T, and this representation factors to $N_{K(\Psi,\mathbb{C})}/T = W$. Call this representation $\tau(\sigma)$. The representation $\tau(\sigma)$ of W controls the occurrence of σ in the principal series, just as in the p-adic case. What is different is (first) that $\tau(\sigma)$ need not be irreducible. Nevertheless, the statements of Theorem 19 make sense, with $\tau(\sigma)$ playing the role of τ. Unfortunately they are false, already for $SL(2, \mathbb{C})$. If we take σ to be the five-dimensional irreducible representation of $SU(2)$, then the Weyl group representation $\tau(\sigma)$ is trivial. The statements of Theorem 19 would therefore predict that the Hermitian form on σ should be positive for all ν. This is not the case: the form is zero for $\nu = \rho$ and 2ρ, and negative between these two points.

On the other hand, most of the proof of Theorem 19 works in the complex case; these difficulties in $SL(2)$ are the only problem. What distinguishes the five-dimensional representation (and larger ones) is that twice a root appears as a weight. Following Reeder,[18] we say that a representation σ of $K(\Psi, \mathbb{C})$ is *small* if $\sigma^T \neq 0$ (so that 0 is a weight of σ) but twice a root is never a weight of σ.

Theorem 20 *Suppose $\Psi = (X, R, X^\vee, R^\vee)$ is a reduced root datum, and R^+ is a set of positive roots for R. Consider principal series for the complex reductive algebraic group $G(\Psi, \mathbb{C})$. Fix a dominant weight $\nu \in \mathfrak{t}(X)$, and assume that $-\nu \in W \cdot \nu$ (so that $\overline{\pi}(\Psi, \nu)$ admits an invariant Hermitian form. Suppose σ is a small irreducible representation of $K(\Psi, \mathbb{C})$, and $\tau(\sigma)$ the representation of W on the zero weight space σ^T.*

1. *The multiplicity $m(\sigma, \nu)$ of the representation σ of $K(\Psi, \mathbb{C})$ in $\overline{\pi}(\Psi, \nu)$ is equal to the rank of the operator $\tau(\sigma)(A(\nu))$.*
2. *The positive part $p(\sigma, \nu)$ of the signature on σ is equal to the number of strictly positive eigenvalues of $\tau(\sigma)(A(\nu))$.*
3. *The negative part $q(\sigma, \nu)$ of the signature on σ is equal to the number of strictly negative eigenvalues of $\tau(\sigma)(A(\nu))$.*

Again the result is due to Barbasch and Vogan. This theorem does not produce any unitary representations, but it can prove that representations are *not* unitary. To do that effectively, we need to get interesting representations of W on the zero weight spaces of small representations of $K(\Psi, \mathbb{C})$. In type A, every representation of W appears in this way (see Reeder[18]). Theorems 16 and 20 therefore imply that

$$P(\text{type } A, \mathbb{C}) \subset P(\text{type } A, F)$$

for any p-adic F. (Actually equality holds, by the calculations of $P(\Psi, F)$ mentioned earlier.)

For groups not of type A, not every Weyl group representation appears in $\tau(\sigma)$ for some small representation σ, so we cannot use Theorem 20 to show that $P(\Psi, \mathbb{C}) \subset P(\Psi, F)$ for F p-adic. Indeed we observed before (24) that this containment is false in types C_2 and G_2; the opposite containment is true, and examples suggest that it may hold in general.

We can now outline a proof of Theorem 16 in the complex case. We may as well assume that the root system R is simple. The complexified adjoint representation Ad of $K(\Psi, \mathbb{C})$ is small, since twice a root is never a root. We may therefore apply Theorem 20 to the representation $\tau(\mathrm{Ad})$ of W; this is the "reflection representation" of W on the complexification of the Lie algebra of T. We are interested in the eigenvalues of the operator $\tau(\mathrm{Ad})(A(\nu))$. To prove Theorem 20, it suffices to show that if there is a simple root δ with $\langle\delta^\vee,\nu\rangle > 1$, then this operator must have at least one negative eigenvalue. Here is a way to do that. Let ν_0 be the sum of the fundamental weights for the simple roots δ and $-w_0\delta$; then ν_0 is dominant, and $w_0\nu_0 = -\nu_0$. We now consider the one-parameter family of self-adjoint operators

$$\tau(\mathrm{Ad})(A(\nu + t\nu_0)),$$

for real $t \geq 0$.

By arranging the reduced expression for w_0 appropriately, it is not too difficult to show that the multiplicity of the eigenvalue 0 of $\tau(\mathrm{Ad})(A(\nu+t\nu_0))$ is independent of t (always for $t \geq 0$). (The key point is that the operators $\tau(\mathrm{Ad})(1 + xs_i)$ appearing in the factorization are invertible for $x > 1$.) It follows by a continuity argument that the number of negative eigenvalues of $\tau(\mathrm{Ad})(A(\nu+t\nu_0))$ is also independent of t. So it suffices to prove that there is a negative eigenvalue for t large. But a very easy argument using the Casimir operator shows that as soon as $|\nu + t\nu_0| > |\rho|$, then the Hermitian form on $\overline{\pi}(\Psi,\nu + t\nu_0)$ must be partly negative on the $K(\Psi,\mathbb{C})$ representation Ad. According to Theorem 20, this means that $\tau(\mathrm{Ad})(A(\nu + t\nu_0))$ has negative eigenvalues, as we wished to show.

The proof of Theorem 16 in the real case proceeds in exactly the same way, using a real analogue of Theorem 20.

Acknowledgments

Supported in part by NSF grant 9721441.

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