Classification of root systems

September 8, 2017

1 Introduction

These notes are an approximate outline of some of the material to be covered on Thursday, April 9; Tuesday, April 14; and Thursday, April 16. They represent a (very slightly) different approach to the material in Chapter 7 of the text. I won't try to make a precise correspondence with the text, but I will try to make the notation match as well as possible with the text. I will also try to include connections with the notion of *root datum* which appeared in the Lie groups class last fall, and which is central to the theory of algebraic groups.

Making that connection requires comparing two extremely elementary ideas.

Definition 1.1. A *Euclidean space* is a finite-dimensional real vector space E endowed with a positive-definite inner product $\langle \cdot, \cdot \rangle$. Any linear map $T \colon V \to V$ has a *transpose* ${}^tT$ defined by

$$\langle Tv, w \rangle = \langle v, {}^tTw \rangle.$$

The map T respects the inner product (that is, T is *orthogonal*) if and only if ${}^tT = T^{-1}$.

If $\alpha \in E$ is any nonzero vector, then the *hyperplane orthogonal to* α is

$$E^\alpha = \{e \in E \mid \langle \alpha, e \rangle = 0\}.$$

There is an orthogonal direct sum decomposition

$$E = \mathbb{R}\alpha \oplus E^\alpha.$$

The *orthogonal reflection in* α is the linear map s_α which is the identity on E^α and -1 on $\mathbb{R}\alpha$. It is given by the formula

$$\begin{aligned} s_\alpha(\xi) &= \xi - 2\frac{\langle \xi, \alpha\rangle}{\langle \alpha, \alpha\rangle}\alpha \\ &= \xi - \langle \xi, \frac{2}{\langle \alpha, \alpha\rangle}\alpha\rangle\alpha. \end{aligned}$$

The reflection s_α respects the inner product and has order 2; so

$$s_\alpha = s_\alpha^{-1} = {}^t s_\alpha.$$

One of the main properties of a reflection is that there are just two eigenvalues: -1 (with multiplicity one) and 1. This property can be expressed without an inner product.

Definition 1.2. Suppose V is a finite-dimensional vector space over a field k of characteristic not equal to two, with dual space V^*. Write

$$\langle \cdot, \cdot \rangle \colon V \times V^* \to k, \quad \langle v, f\rangle = f(v)$$

for evaluation of linear functionals. Any linear map $T \colon V \to V$ has a *transpose* ${}^tT \colon V^* \to V^*$ defined by

$$\langle Tv, f\rangle = \langle v, {}^tTf\rangle.$$

Suppose $\alpha \in V$, $\alpha^\vee \in V^*$, and $\alpha^\vee(\alpha) = 2$. The *hyperplane defined by* $\alpha^\vee$ is

$$V^{\alpha^\vee} = \ker(\alpha^\vee) = \{v \in V \mid \langle v, \alpha^\vee\rangle = 0\}.$$

There is a direct sum decomposition

$$V = k\alpha \oplus V^{\alpha^\vee}.$$

The *reflection in* $(\alpha^\vee, \alpha)$ is the linear map $s_{\alpha^\vee,\alpha}$ which is the identity on $V^{\alpha^\vee}$ and -1 on $k\alpha$. It is given by the formula

$$s_{\alpha^\vee,\alpha}(v) = v - \langle v, \alpha^\vee\rangle\alpha.$$

The reflection $s_{\alpha^\vee,\alpha}$ has order two. Its transpose is the reflection $s_{\alpha,\alpha^\vee}$.

Having dispensed with the inner product, we can even dispense with the field.

Definition 1.3. Suppose L is a lattice: a finitely generated torsion-free abelian group. (That is, L is isomorphic to $\mathbb{Z}^m$ for some nonnegative integer m, called the *rank of* L.) The *dual lattice* is

$$L^* = \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z}),$$

another lattice of rank m. Write

$$\langle \cdot, \cdot \rangle \colon L \times L^* \to \mathbb{Z}, \quad \langle \ell, \lambda \rangle = \lambda(\ell)$$

for evaluation of linear functionals. Any $\mathbb{Z}$-linear map $T \colon L \to L$ has a *transpose* ${}^tT \colon L^* \to L^*$ defined by

$$\langle T\ell, \lambda \rangle = \langle \ell, {}^tT\lambda \rangle.$$

Suppose $\alpha \in L$, $\alpha^\vee \in L^*$, and $\alpha^\vee(\alpha) = 2$. The *hyperplane defined by* $\alpha^\vee$ is

$$L^{\alpha^\vee} = \ker(\alpha^\vee) = \{\ell \in L \mid \langle \ell, \alpha^\vee \rangle = 0\}.$$

The *reflection in* $(\alpha^\vee, \alpha)$ is the $\mathbb{Z}$-linear map

$$s_{\alpha^\vee,\alpha}(\ell) = \ell - \langle \ell, \alpha^\vee \rangle \alpha.$$

Even though L need not be the direct sum of $L^{\alpha^\vee}$ and $\mathbb{Z}\alpha$, the reflection may still be characterized as the unique $\mathbb{Z}$-linear map that is the identity on $L^{\alpha^\vee}$ and minus the identity on α.

The reflection $s_{\alpha^\vee,\alpha}$ has order two. Its transpose is the reflection $s_{\alpha,\alpha^\vee}$.

Here is the main definition from the text.

Definition 1.4. text, Definition 7.1 Suppose E is a Euclidean space (Definition 1.1). A *root system* is a finite collection R of nonzero vectors in a E, subject to the three requirements below.

(R1) R spans E as a vector space.

(R2) For any α and β in R,

$$n_{\alpha\beta} =_{\mathrm{def}} \frac{2\langle \alpha, \beta \rangle}{\langle \beta, \beta \rangle}$$

is an integer.

(R3) For each α in R, the reflection s_α permutes R.

The root system R is called *reduced* if it satisfies in addition

(R4) If α and $c\alpha$ are both in R, then $c = \pm 1$.

The *rank of* R is the dimension of the vector space E.

Suppose $R \subset E$ and $R' \subset E'$ are root systems. An *isomorphism* of R with R' is a linear isomorphism $J \colon E \to E'$ with the property that

$$JR = R', \qquad Js_\alpha = s_{J\alpha}J \quad (\alpha \in R).$$

We do *not* require that J respect the inner products.

Let $\sim$ be the equivalence relation on R generated by the relations

$$\alpha \sim \beta \text{ if } \langle \alpha, \beta \rangle \neq 0.$$

The root system is called *simple* if R is a single equivalence class.

We will recall in Section 2 the connection with semisimple Lie algebras established in Chapter 6 of the text (and in class March 31–April 7).

The axiom (R1) is essentially a historical accident, of no theoretical importance; when the definitions were fixed, the notion of semisimple Lie algebra was primary. Now we might prefer (as the text usually does) to work mostly with reductive Lie algebras. We might call a root system with (R1) a *semisimple root system*, and a root system without (R1) a *reductive root system*. Then it's obvious that any reductive root system is the orthogonal direct sum of a semisimple root system and a Euclidean space with no roots.

Here is a version of this definition without inner products.

Definition 1.5. Suppose V is a finite-dimensional real vector space and V^* is its dual space. A *vector root system* consists of a finite set $R \subset V$; a finite set $R^\vee \subset V^*$; and a bijection $\alpha \mapsto \alpha^\vee$ between R and $R^\vee$, subject to the three requirements below.

(RV1) R spans E as a vector space, and $R^\vee$ spans V^*.

(RV1.5) For any $\alpha \in R$, $\langle \alpha, \alpha^\vee \rangle = 2$.

(RV2) For any $\alpha \in R$ and $\beta \in R^\vee$, $n_{\alpha\beta} =_{\text{def}} \langle \alpha, \beta^\vee \rangle$ is an integer.

(RV3) For each α in R, the reflection $s_{\alpha,\alpha^\vee}$ permutes R, and $s_{\alpha^\vee,\alpha}$ permutes $R^\vee$.

The vector root system R is called *reduced* if it satisfies in addition

(RV4) If α and $c\alpha$ are both in R, then $c = \pm 1$.

An equivalent requirement is

(RV4$^\vee$) If $\alpha^\vee$ and $c\alpha^\vee$ are both in $R^\vee$, then $c = \pm 1$.

The *rank of* $(R, R^\vee)$ is the dimension of the vector space V.

Suppose $(R \subset V, R^\vee \subset V^*)$ and $(R' \subset V', R'^\vee \subset (V')^*)$ are vector root systems. An *isomorphism* between them is a linear isomorphism $J \colon V \to V'$ with the property that

$$JR = R', \qquad {}^tJ^{-1}R^\vee = R'^\vee,$$

and these maps respect the bijections $R \leftrightarrow R^\vee$, $R' \leftrightarrow R'^\vee$.

Let $\sim$ be the equivalence relation on R generated by the relations

$$\alpha \sim \beta \text{ if } \langle \alpha, \beta^\vee \rangle \neq 0.$$

The root system is called *simple* if R is a single equivalence class.

Again, axiom (RV1) is just a historical accident.

So why is this version of the definition worth considering? Examination of Theorem 6.44 in the text shows that the root system for a semisimple Lie algebra in $\mathfrak{h}^*$ is really constructed as a vector root system with the roots $\alpha \in \mathfrak{h}^*$ and the coroots $\alpha^\vee = h_\alpha \in \mathfrak{h}$ (constructed in Lemma 6.42); the inner products are just added decoration.

Having disposed of the inner product, we can even dispose of the real numbers entirely.

Definition 1.6. Suppose L is a lattice and L^* is its dual lattice. A *root datum* consists of a finite set $R \subset L$; a finite set $R^\vee \subset L^*$; and a bijection $\alpha \mapsto \alpha^\vee$ between R and $R^\vee$, subject to the three requirements below.

(RD1.5) For any $\alpha \in R$, $\langle \alpha, \alpha^\vee \rangle = 2$.

(RD3) For each α in R, the lattice reflection $s_{\alpha,\alpha^\vee}$ (Definition 1.3) permutes R, and $s_{\alpha^\vee,\alpha}$ permutes $R^\vee$.

The root datum is called *reduced* if it satisfies in addition

(RD4) If α and $c\alpha$ are both in R, then $c = \pm 1$.

An equivalent requirement is

(RD4$^\vee$) If $\alpha^\vee$ and $c\alpha^\vee$ are both in $R^\vee$, then $c = \pm 1$.

The *rank of* $(R, R^\vee)$ is the rank of the lattice L. The *root lattice of* R is the sublattice $Q(R) \subset L$ generated by R. The *semisimple rank* is the rank of $Q(R)$.

Suppose $(R \subset L, R^\vee \subset L^*)$ and $(R' \subset L', R'^\vee \subset (L')^*)$ are root data. An *isomorphism* between them is a lattice isomorphism $\Lambda\colon L \to L'$ with the property that

$$\Lambda R = R', \qquad {}^t\Lambda^{-1} R^\vee = R'^\vee,$$

and these maps respect the bijections $R \leftrightarrow R^\vee$, $R' \leftrightarrow R'^\vee$.

Let $\sim$ be the equivalence relation on R generated by the relations

$$\alpha \sim \beta \text{ if } \langle \alpha, \beta^\vee \rangle \neq 0.$$

The root system is called *simple* if R is a single equivalence class.

This definition (due to Grothendieck and his school in the 1960s) was formulated when the primary role of *reductive* groups had become clear; so there is no "semisimplicity" axiom analogous to (R1) or (RV1). (How might you formulate one if you wanted it?) In this version of the definition the second (integrality) axiom has entirely disappeared: it is embedded in the fact that we are using a $\mathbb{Z}$-valued pairing. The main axioms are (RD1.5) (saying that root reflections can be defined) and (RD3) (saying that they permute roots and coroots). This is the real heart of the matter.

Definition 1.7. A *diagram* is a finite graph in which each edge is either single, or double with an arrow, or triple with an arrow.

The goal of Chapter 7 in the text, and of these notes, is

Theorem 1.8. *Attached to each root system (or vector root system, or root datum) is a diagram Γ. The number of connected components of Γ is the number of equivalence classes in R. Each connected component of Γ appears in Table 1.*

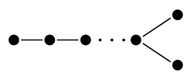
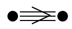
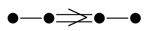
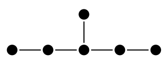
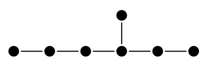
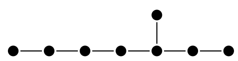
2 Connection with Lie algebras

In this section we recall the construction of a root system from a semisimple Lie algebra given in Chapter 6 of the text. We will also state Grothendieck's version of the classification of compact Lie groups (as motivation for the definition of root datum), and the classification of reductive algebraic groups (for the same reason).

3 Root systems and root data

In this section we will describe how to relate the three versions of root system in the introduction.

Table 1: Dynkin diagrams

type	diagram	$\#R$	$\#W$
A_n $(n \geq 1)$		$n^2 + n$	$(n+1)!$
B_n $(n \geq 2)$		$2n^2$	$2^n \cdot n!$
C_n $(n \geq 3)$		$2n^2$	$2^n \cdot n!$
D_n $(n \geq 4)$		$2n^2 - 2n$	$2^{n-1} \cdot n!$
G_2		12	12
F_4		48	1152
E_6		72	$2^7 \cdot 3^4 \cdot 5 = 51{,}840$
E_7		126	$2^{10} \cdot 3^4 \cdot 5 \cdot 7 = 2{,}903{,}040$
E_8		240	$2^{14} \cdot 3^5 \cdot 5^2 \cdot 7 = 696{,}729{,}600$

4 The Weyl group

In this section we define the Weyl group of a root system. This is an extraordinarily important finite group. Much of Lie theory is dedicated to translating difficult problems into questions about the Weyl group. To understand how this looks, consider the simple Lie algebra $\mathfrak{sl}(n)$ of trace zero $n \times n$ matrices. The Weyl group is the symmetric group S_n. You may know that the irreducible representations of the symmetric group are indexed by partitions of n; and you should certainly know that conjugacy classes of $n \times n$ nilpotent matrices (equivalently, nilpotent elements of $\mathfrak{sl}(n)$) are also indexed by partitions of n. This is not a coincidence. At a deeper level, some of the most complicated questions about representation theory for any semisimple Lie algebra $\mathfrak{g}$ are answered by the "Kazhdan-Lusztig polynomials;" and these polynomials are constructed by elementary combinatorics on the Weyl group.

Definition 4.1. Suppose R is a root system (Definition 1.4). The *Weyl group of* R is the subgroup $W(R)$ of automorphisms of E generated by the reflections $\{s_\alpha \mid \alpha \in R\}$. By (R3), the elements of $W(R)$ permute R. Since R spans E, we may identify $W(R)$ with the corresponding group of permutations of R; so $W(R)$ is finite.

Here is the same definition in the language of vector root systems.

Definition 4.2. Suppose $(R, R^\vee)$ is a vector root system (Definition 1.5). The *Weyl group of* R is the subgroup $W(R)$ of automorphisms of V generated by the reflections $\{s_{\alpha^\vee,\alpha} \mid \alpha \in R\}$. By (RV3), the elements of $W(R)$ permute R. Since R spans V, we may identify $W(R)$ with the corresponding group of permutations of R; so $W(R)$ is finite.

Taking inverse transpose identifies $W(R)$ with the Weyl group $W(R^\vee)$ of automorphisms of V^* generated by the reflections $\{s_{\alpha,\alpha^\vee} \mid \alpha^\vee \in R^\vee\}$.

Finally, here is the definition in the language of root data.

Definition 4.3. Suppose $(R \subset L, R^\vee \subset L^*)$ is a root datum (Definition 1.6). The *Weyl group of* R is the subgroup $W(R)$ of automorphisms of L generated by the reflections $\{s_{\alpha^\vee,\alpha} \mid \alpha \in R\}$. By (RD3), the elements of $W(R)$ permute R.

Taking inverse transpose identifies $W(R)$ with the Weyl group $W(R^\vee)$ of automorphisms of L^* generated by the reflections $\{s_{\alpha,\alpha^\vee} \mid \alpha^\vee \in R^\vee\}$.

In this setting the finiteness of $W(R)$ is not quite so obvious.

5 Rank two

In this section we analyze root systems of rank two; that is, the possible behavior of two roots α and β is an arbitrary root system. The case of root data is the most subtle, so we will concentrate on that, leaving the simplifications possible for root systems to the reader.

Here is the underlying algebra fact.

Lemma 5.1. *Suppose A is a 2×2 matrix of integers having some finite order m; that is, that $A^m = I$. There are the following possibilities.*

1. *Both eigenvalues of A are $+1$, $\operatorname{tr} A = 2$, and $A = I$.*
2. *Both eigenvalues of A are -1, $\operatorname{tr} A = -2$, and $A = -I$.*

3. *The eigenvalues of A are $+1$ and -1, $\operatorname{tr} A = 0$, and $A^2 = I$.*

4. *The eigenvalues of A are $e^{\pm 2\pi i/3}$, $\operatorname{tr} A = -1$, and $m = 3$.*

5. *The eigenvalues of A are $e^{\pm 2\pi i/4}$, $\operatorname{tr} A = 0$, and $m = 4$.*

6. *The eigenvalues of A are $e^{\pm 2\pi i/6}$, $\operatorname{tr} A = +1$, and $m = 6$.*

Proof. The first three cases are the only possibilities with real eigenvalues; so we may assume that $m > 2$, and that the eigenvalues are $e^{\pm 2\pi i k/m}$, with k an integer relatively prime to m. Then $\operatorname{tr}(A) = 2\cos(2\pi k/m)$. Since A has integer entries, this trace is an integer. Since it is also twice a cosine, and not ± 2, it must be -1, 0, or 1. $\square$

We fix throughout this section a root datum

$$(R \subset L, R^\vee \subset L^*) \tag{5.2}$$

as in Definition 1.6.

Lemma 5.3. *Suppose α and β are nonproportional elements of R. Then there are four possibilities.*

1. *$\langle \alpha, \beta^\vee \rangle = 0$. In this case also $\langle \beta, \alpha^\vee \rangle = 0$. The two reflections $s_{\alpha^\vee,\alpha}$ and $s_{\beta^\vee,\beta}$ commute; their product acts by $+1$ on the common kernel of $\alpha^\vee$ and $\beta^\vee$, and by -1 on the span of α and β.*

Otherwise $\langle \alpha, \beta^\vee \rangle = n \neq 0$ is a nonzero integer. In this case $\langle \beta, \alpha^\vee \rangle = n'$ is a nonzero integer of the same sign, and $1 \le nn' \le 3$. Possibly after replacing β by $-\beta$, we may assume that n and n' are both negative. *Possibly after interchanging α and β, we may assume that $n \le n'$.*

2. *If $n = n' = 1$, then $s_{\alpha^\vee,\alpha} s_{\beta^\vee,\beta}$ has order three. The (root, coroot) pairs include*

$$(\alpha, \alpha^\vee), \quad (\beta, \beta^\vee) \quad (s_{\alpha^\vee,\alpha}(\beta) = \alpha + \beta, \alpha^\vee + \beta^\vee)$$

and their negatives, constituting a subsystem of type A_2.

3. *If $n = 1$ and $n' = 2$, then $s_{\alpha^\vee,\alpha} s_{\beta^\vee,\beta}$ has order four. The (root,coroot) pairs include*

$$\begin{array}{cc} \alpha & \alpha^\vee \\ \beta & \beta^\vee \\ s_{\beta^\vee,\beta}(\alpha) = \alpha + \beta & s_{\beta,\beta^\vee}(\alpha^\vee) = \alpha^\vee + 2\beta^\vee \\ s_{\alpha^\vee,\alpha}(\beta) = 2\alpha + \beta & s_{\alpha,\alpha^\vee}(\beta^\vee) = \alpha^\vee + \beta^\vee \end{array}$$

and their negatives, constituting a subsystem of type B_2.

4. If $nn' = 3$, then $s_{\alpha^\vee,\alpha}s_{\beta^\vee,\beta}$ has order six. The (root,coroot) pairs include

$$\begin{array}{cc}
\alpha & \alpha^\vee \\
\beta & \beta^\vee \\
s_{\beta^\vee,\beta}(\alpha) = \alpha+\beta & s_{\beta,\beta^\vee}(\alpha^\vee) = \alpha^\vee + 3\beta^\vee \\
s_{\alpha^\vee,\alpha}(\beta) = 3\alpha+\beta & s_{\alpha,\alpha^\vee}(\beta^\vee) = \alpha^\vee+\beta^\vee \\
s_{\alpha^\vee,\alpha}(\alpha+\beta) = 2\alpha+\beta & s_{\alpha,\alpha^\vee}(\alpha^\vee+3\beta^\vee) = 2\alpha^\vee+3\beta^\vee \\
s_{\beta^\vee,\beta}(3\alpha+\beta) = 3\alpha+2\beta & s_{\beta,\beta^\vee}(\alpha^\vee+\beta^\vee) = \alpha^\vee+2\beta^\vee
\end{array}$$

and their negatives, constituting a subsystem of type G_2.

Corollary 5.4. *Suppose α and β are nonproportional roots. If*

$$\langle \alpha, \beta^\vee\rangle = m > 0,$$

then the m elements

$$\alpha - \beta, \cdots, \alpha - m\beta = s_{\beta^\vee,\beta}(\alpha)$$

are all roots. If

$$\langle \alpha, \beta^\vee\rangle = -m < 0,$$

then

$$\alpha + \beta, \cdots, \alpha + m\beta = s_{\beta^\vee,\beta}(\alpha)$$

are all roots.

6 Positive root systems and Dynkin diagrams

In this section we define positive roots and simple roots for a root system, and use them to define the Dynkin diagram of a root system. At the same time we show how to recover all the roots from the Dynkin diagram.

Definition 6.1. Suppose $R \subset E$ is a root system. An element $\lambda \in E$ is called *regular* if λ is not orthogonal to any root. A *system of positive roots for R* is a subset $R^+ \subset R$ so that for some regular element λ, we have

$$R^+ = \{\beta \in R \mid (\beta, \lambda) > 0\}.$$

The set of *simple roots for R^+* consists of the positive roots that cannot be written as sums of positive roots:

$$\Pi = \Pi(R^+) = \{\gamma \in R^+ \mid \gamma \neq \alpha + \beta \quad (\alpha, \beta \in R^+)\}.$$

The collection of nonregular elements of E is a finite union of hyperplanes, and therefore cannot be equal to E. So regular elements exist, and positive root systems exist.

We will leave to the reader the definition of positive root systems in the vector root case, and pass directly to root data.

Definition 6.2. Suppose $(R \subset L, R^\vee \subset L^*)$ is a root datum. An element $\lambda \in L^*$ is called *regular* if λ does not vanish on any root. A *system of positive roots for* R is a subset $R^+ \subset R$ so that for some regular element λ, we have

$$R^+ = \{\beta \in R \mid (\beta, \lambda) > 0\}.$$

The set of *simple roots for* R^+ consists of the positive roots that cannot be written as sums of positive roots:

$$\Pi = \Pi(R^+) = \{\gamma \in R^+ \mid \gamma \neq \alpha + \beta \quad (\alpha, \beta \in R^+)\}.$$

The kernel in L^* of a nonzero element of L (such as α) is a sublattice of rank one less. The set of nonregular elements of L^* is therefore a finite union of sublattices of lower rank. A lattice is never the union of sublattices of lower rank; so regular elements exist, and positive root systems exist.

Here are some of the basic facts about positive roots and simple roots.

Proposition 6.3. *Suppose R is a root system (or vector root system, or root datum). Then there is a set of positive roots for R. Fix such a set R^+, and let Π be the corresponding set of simple roots.*

1. *The root system R is the disjoint union of R^+ and $-R^+$.*
2. *If $\alpha, \beta \in R^+$, $p, q \in \mathbb{N}$, and $p\alpha + q\beta \in R$, then $p\alpha + q\beta \in R^+$.*
3. *For every positive root β there are expressions*

$$\beta = \sum_{\alpha \in \Pi} m_\alpha \alpha, \qquad \beta^\vee = \sum_{\alpha \in \Pi} m_{\alpha^\vee} \alpha^\vee,$$

   with each m_α and $m_\alpha^\vee$ a non-negative integer.

4. *Suppose $\beta \neq \alpha$ are positive roots, and α is simple. If $\beta - \alpha$ is a root, then it is positive.*
5. *Suppose $\alpha_1 \neq \alpha_2$ are simple roots. Then $\langle \alpha_1, \alpha_2^\vee \rangle \leq 0$; it is nonzero if and only if $\langle \alpha_2, \alpha_1^\vee \rangle$ is also nonzero. In this case one of the numbers is -1, and the other is -1, -2, or -3.*

6. *Suppose β is a positive root, and β is not simple. Then there is a simple root α so that*
$$\langle \beta, \alpha^\vee \rangle = m > 0$$
is a strictly positive integer, and
$$s_\alpha(\beta) = \beta - m\alpha \in R^+.$$

Before we embark on the proof of Proposition 6.3, we record the definition of Dynkin diagram that it provides.

Definition 6.4. Suppose R is a root system (or vector root system, or root datum), R^+ is a system of positive roots, and Π the corresponding set of simple roots (Definition 6.1. The *Dynkin diagram* of R (Definition 1.7) is the graph with vertex set Π. Two simple roots α_1 and α_2 are joined if and only if $\langle \alpha_1, \alpha_2^\vee \rangle < 0$. In this case they are joined by a bond of multiplicity
$$\langle \alpha_1, \alpha_2^\vee \rangle \langle \alpha_2, \alpha_1^\vee \rangle$$
(which is 1, 2, or 3); in the double and triple cases, there is an arrow pointing to α_i if $\langle \alpha_i, \alpha_j^\vee \rangle = -1$. In the root system case (that is, with a Euclidean structure) the arrow points to the root that is strictly shorter.

The Dynkin diagram therefore records all pairings between simple roots and coroots, and so records formulas for the action of the simple reflections on these elements.

Next, we notice that Proposition 6.3 gives a method to reconstruct all of R from the Dynkin diagram. Here is the procedure. I will phrase it for the case of root data; the modifications for the other cases are simple.

Begin with the set $R_1 = \Pi$ (the vertices of the diagram). We are going to construct additional sets of positive roots R_2, R_3, and so on. We always construct R_{i+1} from R_i by the same procedure. For each element β in R_i, list all the simple roots α so that
$$\langle \beta, \alpha^\vee \rangle = m < 0$$
For each such root, we introduce the element
$$s_{\alpha^\vee, \alpha}(\beta) = \beta + m\alpha \in R_{i+1}.$$
It is a root by (RD3) of Definition 1.6, and positive by Proposition 6.3(2). Therefore all the sets R_i are contained in R^+. We claim that
$$\bigcup_i R_i = R^+.$$

Now Proposition 6.3(6) allows one to prove that β belongs to some R_i by induction on $\langle\beta,\lambda\rangle$, with λ a regular element defining R^+.

Example 6.5. Let $E = \{v \in \mathbb{R}^3 \mid \sum v_i = 0\}$, with the usual inner product. It turns out that the two elements

$$\alpha_1 = (1,-1,0), \qquad \alpha_2 = (-2,1,1)$$

are a set of simple roots for a root system. Let's construct the rest of the positive roots. To build R_2, we begin with $\alpha_1 \in R_1$ and find all the simple roots having negative inner product with it. There is only one, namely α_2; the corresponding integer is

$$2\langle\alpha_1,\alpha_2\rangle/\langle\alpha_2,\alpha_2\rangle = 2(-3)/6 = -1,$$

so we get

$$s_{\alpha_2}(\alpha_1) = \alpha_1 + \alpha_2 = (-1,0,1) \in R_2.$$

A similar calculation with α_2 leads to

$$s_{\alpha_1}(\alpha_2) = \alpha_2 + 3\alpha_1 = (1,-2,1) \in R_2.$$

Next we calculate R_3. The element $(-1,0,1) \in R_2$ has negative inner product only with the simple root α_1, so we get

$$s_{\alpha_1}(-1,0,1) = (-1,0,1) + \alpha_1 = (0,-1,1) \in R_3.$$

Similarly $(1,-2,1)$ leads to

$$s_{\alpha_2}(1,-2,1) = (1,-2,1) + \alpha_2 = (-1,-1,2) \in R_3.$$

Both roots in R_3 have non-negative inner product with both simple roots, so R_4 and later sets are empty. Therefore

$$R^+ = \{(1,-1,0),(-2,1,1),(-1,0,1),(1,-2,1),(0,-1,1),(-1,-1,2)\}.$$

The full root system includes the negatives of these six; it is the root system of type G_2.

In the case of E_8, a calculation like the one just given for G_2 has 29 steps; that is, R_{30} is empty. It produces explicit formulas for the 120 positive roots as sums of simple roots. Such a calculation can be done by hand without too much difficulty, and of course is utterly trivial for a computer.

Proof of Proposition 6.3. We will work in the case of a root datum; the other two cases are similar but simpler.

Fix a regular element $\lambda \in L^*$ defining R^+:

$$\langle \alpha, \lambda \rangle > 0 \qquad (\alpha \in R^+). \tag{6.6}$$

Parts (1) and (2) of the proposition are immediate. Let $\epsilon > 0$ be the smallest possible value of a pairing product $\langle \alpha, \lambda \rangle$, with $\alpha \in R^+$; this is a strictly positive number (actually a positive integer in the root datum case) since R is finite. To prove (3), find a positive integer N so that

$$(N+1)\epsilon > \langle \beta, \lambda \rangle \geq N\epsilon;$$

we prove (3) by induction on N. If β is simple there is nothing to prove; so suppose $\beta = \beta_1 + \beta_2$ with $\beta_i \in R^+$. Define N_i for β_i as above. Since $\langle \beta_i, \gamma \rangle \geq \epsilon$, we have $N_i < N$. By inductive hypothesis, we can write

$$\beta_i = \sum_{\alpha \in \Pi} m_{\alpha,i} \alpha.$$

Taking $m_\alpha = m_{\alpha,1} + m_{\alpha,2}$ gives (3).

For (4), if $\beta - \alpha$ is negative, then $\alpha - \beta$ is positive, and the expression $\alpha = (\alpha - \beta) + \beta$ contradicts the assumed simplicity of α.

For part (5), suppose $\langle \alpha_1, \alpha_2^\vee \rangle > 0$. We deduce from Corollary 5.4 that $\langle \alpha_2, \alpha_1^\vee \rangle > 0$, and that $\alpha_1 - \alpha_2$ is a root. By (4), this root is both positive and negative. This contradiction proves that $\langle \alpha_1, \alpha_2 \rangle \leq 0$. The remaining assertions are in Lemma 5.3.

For (6), write $\beta^\vee = \sum_{\delta \in \Pi} m_{\delta^\vee} \delta^\vee$ as in (3).

$$2 = \langle \beta, \beta^\vee \rangle = \sum_{\delta \in \Pi} m_{\delta^\vee} \langle \beta, \delta^\vee \rangle.$$

So at least one summand is strictly positive; that is, there is a simple root α with $m_{\alpha^\vee} > 0$ and $\langle \beta, \alpha^\vee \rangle = m > 0$. By Corollary 5.4,

$$\beta, \beta - \alpha, \ldots, s_{\alpha^\vee,\alpha}(\beta) = \beta - m\alpha$$

are all roots; so by (4) they are all positive. $\square$

Corollary 6.7. *Suppose R is a root system (or vector root system, or root datum) with positive root system R^+, and simple roots Π (Proposition 6.3). Define*

$$2\rho = \sum_{\beta \in R^+} \beta = \sum_{\alpha \in \Pi} r_\alpha \alpha, \quad 2\rho^\vee = \sum_{\beta \in R^+} \beta^\vee = \sum_{\alpha \in \Pi} r_{\alpha^\vee} \alpha^\vee;$$

here r_α and $r_{\alpha^\vee}$ are strictly positive integers.

1. *If $\alpha \in \Pi$, then the simple reflection $s_{\alpha^\vee,\alpha}$ preserves $R^+ \backslash \alpha$.*

2. *If $\alpha \in \Pi$,*
$$\langle \alpha, 2\rho^\vee \rangle = 2, \qquad \langle 2\rho, \alpha^\vee \rangle = 2.$$

3. *The element $2\rho^\vee \in L^*$ takes strictly positive (even integer) values on R^+; so it is regular, and may be used as the element defining R^+.*

4. *The element $2\rho \in L$ takes strictly positive even integer values on $(R^+)^\vee$; so these coroots form a positive system.*

5. *The set Π is linearly independent; it is a basis for the vector space E (in the root system case) or V (in the vector root case) or for the root lattice $Q(R) \subset L$ (in the root datum case).*

Proof. For (5), suppose that $\sum_{\alpha\in\Pi} p_\alpha \alpha = 0$, but that the p_α are not all zero. We may assume that some $p_{\alpha_0} > 0$.

Suppose first that all of the p_α are non-negative. Pairing with the element $\lambda \in L^*$ defining R^+, we get
$$0 = \sum p_\alpha \langle \alpha, \lambda \rangle.$$
Since the terms on the right are all non-negative and one is positive, this is impossible.

We conclude that some of the p_α are strictly negative. Define
$$\Pi_A = \{\alpha \in \Pi \mid p_\alpha > 0\}, \qquad \Pi_B = \{\alpha \in \Pi \mid p_\alpha < 0\}.$$
These are non-empty disjoint subsets of Π. We have
$$\sum_{\alpha\in\Pi_A} p_\alpha \alpha = \sum_{\delta\in\Pi_B} (-p_\delta)\delta. \tag{6.8}$$
By the first part of the argument, all the coefficients on each side are non-negative, and one p_{α_0} on the left is strictly positive.

Now define R_A to be the root datum generated by Π_A (with roots obtained by applying simple reflections from this subset repeatedly to Π_A) and define
$$\rho_A^\vee = \sum_{\alpha\in\Pi_A} r_{\alpha^\vee,A}\alpha^\vee$$
for this root system as above, with all $r_{\alpha^\vee,A}$ strictly positive integers. Applying $\rho_A^\vee$ to the left side of (6.8) gives
$$2\sum_{\alpha\in\Pi_A} p_\alpha \geq 2p_{\alpha_0} > 0.$$

Applying $\rho_A^\vee$ to the right side of (6.8) gives

$$\sum_{\alpha \in \Pi_A, \delta \in \Pi_B} (-p_\delta) r_{\alpha^\vee, A} \langle \delta, \alpha^\vee \rangle.$$

Because Π_A and Π_B are disjoint, all the factors $\langle \delta, \alpha^\vee \rangle$ are nonpositive; and the coefficients are nonnegative. So the right side is nonpositive, a contradiction. The conclusion is that no linear dependence relation among the roots in Π is possible, as we wished to show. $\square$

7 Coxeter graphs and the classification

In this section we introduce the "labelled extended Dynkin diagram" for a simple root system. In the simply laced case, we prove that any labelled extended diagram is a "Coxeter graph," and classify the Coxeter graphs. This will complete the proof of Theorem 1.8 in the simply laced case.

Definition 7.1. Suppose R is a simple root system with positive roots R^+ and simple roots Π. A positive root β is called *highest* if $\beta + \alpha$ is not a root for any simple root α. Similarly, a negative root γ is called *lowest* if $\gamma - \alpha$ is not a root for any simple root α.

If γ is a lowest root, then the *extended Dynkin diagram* is constructed in essentially the same way as the Dynkin diagram, adding the additional vertex γ:

$$\Pi^* = \Pi \cup \{\gamma\}.$$

For the definition of edges, the only new case is that if $\gamma = -\alpha$ (which can happen only in type A_1). If

$$\langle \alpha, \gamma^\vee \rangle = \langle \gamma, \alpha^\vee \rangle = -2,$$

then we join α to γ by two single bonds (not a double bond).

The special vertex γ may be depicted with a star.

According to Proposition 6.3(3)

$$\gamma = -\sum_{\alpha \in \Pi} m_\alpha \alpha$$

for nonnegative integers m_α; in fact the "lowest" condition makes all $m_\alpha > 0$. We define $m_\gamma = 1$, so that

$$\sum_{\alpha' \in \Pi^*} m_{\alpha'} \alpha' = 0. \tag{7.2}$$

The *labelled extended Dynkin diagram* has the positive integer $m_{\alpha'}$ labelling each vertex α'.

Definition 7.3. A *Coxeter graph* is a connected graph subject to the following requirements.

1. Each vertex α' is labelled by a positive integer $m_{\alpha'}$.
2. There is a distinguished vertex γ labelled 1.
3. If α is any vertex, then

$$\sum_{\alpha' \text{ adjacent to } \alpha} m_{\alpha'} = 2m_\alpha.$$

Proposition 7.4. *Every finite Coxeter graph is either A_n^* (for $n \geq 1$) or D_n* (for $n \geq 4$) or E_n^* (for $n = 6, 7, 8$).*

This could be proved by clever elementary school students.

Proposition 7.5. *The labelled extended Dynkin diagram of a simply laced simple root system is a finite Coxeter graph. Conversely, any finite Coxeter graph is the labelled extended Dynkin diagram of a simply laced simple root system.*

The first statement is a simple exercise in the definitions. The interesting property (3) in the definition of Coxeter graph comes by applying (7.2) to the coroot $\alpha^\vee$.

The converse is more difficult, but not a great deal. One way to prove it uses the following lemma from Bourbaki's Lie algebra book (Exercise 7 for Chapter V, §3).

Lemma 7.6. *Suppose $A = (a_{ij})$ is a symmetric $n \times n$ real matrix with $a_{ij} \leq 0$ whenever $i \neq j$. Introduce an equivalence relation $\sim$ on $\{1, \ldots, n\}$ by requiring $i \sim j$ whenever $a_{ij} \neq 0$; and assume that there is just one equivalence class. Consider the quadratic form*

$$Q(x) = x^t A x = \sum_{i,j} a_{ij} x_i x_j.$$

Assume that there are real numbers $\zeta_i > 0$ so that

$$\sum_i \zeta_i a_{ij} = 0 \qquad (1 \leq j \leq n).$$

Then Q is positive semidefinite, with radical spanned by $\zeta \in \mathbb{R}^n$.