The River for $x^2 - 7y^2$

I explained in class that you could solve Pell's equation $x^2-Ny^2 = 1$ by following the river for the quadratic form x^2-Ny^2. The river begins on the segment between the vectors $(1, 0)$ and $(0, 1)$, where the form takes the values 1 and $-N$. Since the river is periodic, you must by following it eventually reach another segment with one bank labelled by the value 1. This means exactly that the primitive vector (x, y) in that region is a solution of Pell's equation.

When I tried to do this for the case $N = 7$, I made a series of arithmetic errors. The point of these notes is to correct the errors. Recall that near any segment, Conway's map for a quadratic form looks like

$$\begin{array}{ccccc} & \diagdown & A & \diagup & \\ A+C-B & & \rule[0.5ex]{3em}{0.4pt} & & A+C+B \\ & \diagup & C & \diagdown & \end{array}$$

Here the top region corresponds to a primitive vector v_1, the bottom region to a primitive vector v_2, the left one to $v_1 - v_2$, and the right one to $v_1 + v_2$. The numbers shown in the picture are values of the form:

$$Q(v_1) = A, \quad Q(v_2) = C, \quad Q(v_1 - v_2) = A + C - B, \quad Q(v_1 + v_2) = A + C + B.$$

The discriminant D of the form is $B^2 - 4AC$. The point of these formulas is that if you know the numbers and vectors attached to all but the rightmost region, then you can calculate the number and vector attached to the rightmost region. Exactly one of the two diagonal segments on the right will be the next segment of the river. For that segment, you know everything for all but the rightmost region, etc.

The map for $x^2 - 7y^2$ begins with

$$\begin{array}{ccccc} & \diagdown & 1 & \diagup & \\ -6 & & \rule[0.5ex]{3em}{0.4pt} & & \\ & \diagup & -7 & \diagdown & \end{array}$$

The upper and lower regions correspond to the vectors $v_1 = (1, 0)$ and $v_2 = (0, 1)$ respectively, and the one on the left to $(1, -1)$. It follows that the region on the right is labelled -6, and corresponds to the vector $(1, 1)$.

The next segment of the river is therefore the upper diagonal on the right; moving it to center stage, we get the picture

$$\begin{array}{ccccc} & \diagdown & 1 & \diagup & \\ -7 & & \rule[0.5ex]{3em}{0.4pt} & & \\ & \diagup & -6 & \diagdown & \end{array}$$

with upper and lower regions corresponding to $v_1 = (1, 0)$ and $v_2 = (1, 1)$. It follows that the righthand region is labelled -3, and corresponds to the vector $v_1 + v_2 = (2, 1)$.

The calculation proceeds in this way. Here is a table in which each row begins with a segment of the river, and ends with the three vectors v_1, v_2, and v_1+v_2. The first row is the one described at the beginning; each subsequent row is calculated from its predecessor as described in the last paragraph.

$$\begin{array}{ccccc} & \diagdown & 1 & \diagup & \\ -6 & & \overline{} & & -6 \\ & \diagup & -7 & \diagdown & \end{array} \qquad\qquad (1,0) \quad (0,1) \quad (1,1)$$

$$\begin{array}{ccccc} & \diagdown & 1 & \diagup & \\ -7 & & \overline{} & & -3 \\ & \diagup & -6 & \diagdown & \end{array} \qquad\qquad (1,0) \quad (1,1) \quad (2,1)$$

$$\begin{array}{ccccc} & \diagdown & 1 & \diagup & \\ -6 & & \overline{} & & 2 \\ & \diagup & -3 & \diagdown & \end{array} \qquad\qquad (1,0) \quad (2,1) \quad (3,1)$$

$$\begin{array}{ccccc} & \diagdown & 2 & \diagup & \\ 1 & & \overline{} & & -3 \\ & \diagup & -3 & \diagdown & \end{array} \qquad\qquad (3,1) \quad (2,1) \quad (5,2)$$

$$\begin{array}{ccccc} & \diagdown & 2 & \diagup & \\ -3 & & \overline{} & & 1 \\ & \diagup & -3 & \diagdown & \end{array} \qquad\qquad (2,1) \quad (5,2) \quad (8,3)$$

At last we have come to another region labelled 1 on the river: the right region of the last diagram, corresponding to the vector $(8,3)$. Indeed

$$8^2 - 7 \cdot 3^2 = 64 - 63 = 1,$$

so $(8,3)$ is a solution of Pell's equation.

To see the periodicity in the river, we need to compute three more segments. The upper and lower values on those next three segments are $(1,-3)$, $(1,-6)$, and $(1,-7)$. From there the river repeats; the period is equal to 7.