Conjugacy classes and group representations

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Outline

What's representation theory about?

Abstract symmetry and groups

Conjugacy classes

Representation theory

Symmetric groups and partitions

Matrices and eigenvalues

Conclusion

The talk in one slide

Three topics. . .

1. GROUPS: abstract way to think about symmetry.
2. CONJUGACY CLASSES: organizing group elements.
3. REPRESENTATIONS: linear algebra and group theory.

Representations of $G \overset{\text{crude}}{\leftrightsquigarrow}$ conjugacy classes in G.

Better: relation is like duality for vector spaces.

dim(reps), size(conj classes) $\leftrightsquigarrow$ noncommutativity.

dim(representation) $\overset{??}{\leftrightsquigarrow}$ $(\text{size})^{1/2}$(conjugacy class).

Talk is about examples of all these things.

Two cheers for linear algebra

My favorite mathematics is linear algebra.

Complicated enough to describe interesting stuff.

Simple enough to calculate with.

Linear map $T\colon V \to V \rightsquigarrow$ eigenvalues, eigenvectors.

First example: $V =$ fns on $\mathbb{R}$, $S =$ chg vars $x \mapsto -x$.

Eigvals: ± 1. Eigenspace for $+1$: even fns (like $\cos(x)$).
Eigenspace for -1: odd functions (like $\sin(x)$).

Linear algebra says: to study sign changes in x, write fns using even and odd fns.

Second example: $V =$ functions on $\mathbb{R}$, $T = \frac{d}{dx}$.

Eigenvals: $\lambda \in \mathbb{C}$. Eigenspace for λ: multiples of $e^{\lambda x}$.

Linear algebra says: to study $\frac{d}{dx}$, write functions using exponentials $e^{\lambda x}$.

The third cheer for linear algebra

Best part about linear algebra is noncommutativity...

Third example: $V =$ fns on $\mathbb{R}$, $S = (x \mapsto -x)$, $T = \frac{d}{dx}$.

S and T don't commute; can't diagonalize both.

Only common eigenvectors are constant fns.

Representation theory idea: look at smallest subspaces preserved by both S and T.

$$W_{\pm\lambda} = \langle \underbrace{e^{\lambda x}, e^{-\lambda x}}_{\text{eigenfns of } d/dt} \rangle = \langle \underbrace{\cosh(\lambda x)}_{\text{even}}, \underbrace{\sinh(\lambda x)}_{\text{odd}} \rangle.$$

These two bases of $W_{\pm\lambda}$ are good for different things.

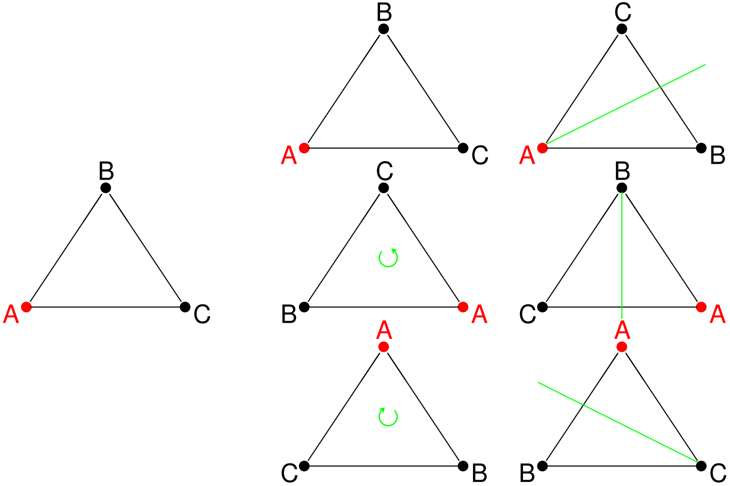
First for solving diff eqs, second for describing bridge cable.

No one basis is good for everything.

Six symmetries of a triangle

Basic idea in mathematics is symmetry.

A symmetry of something is a way of rearranging it so that nothing you care about changes.



Composing symmetries

What you can do with symmetries is compose them.

If g and h are symmetries, so is

$$g \circ h =_{\text{def}} \text{first do } h, \text{ then do } g$$

$\triangle$ example: if $r_{240} =$ rotate 240°, $r_{120} =$ rotate 120°,

$$r_{240} \circ r_{120} = \text{rotate } (240^\circ + 120^\circ) = \text{do nothing} = r_0.$$

Harder: if $r_{240} =$ rotate 240°, $s_A =$ reflection fixing A,

$$r_{240} \circ s_A = \text{exchange } B \text{ and } C, \text{ then } A \to B \to C \to A$$
$$= (A \to B, \quad B \to A, \quad C \to C) = s_C.$$

Composition law for triangle symmetries

We saw that the triangle has six symmetries:

$r_0 \quad r_{120} \quad r_{240}$ rotations

$s_A \quad s_B \quad s_C$ reflections.

Here is how you compose them.

$\circ$	r_0	r_{120}	r_{240}	s_A	s_B	s_C
r_0	r_0	r_{120}	r_{240}	s_A	s_B	s_C
r_{120}	r_{120}	r_{240}	r_0	s_B	s_C	s_A
r_{240}	r_{240}	r_0	r_{120}	s_C	s_A	s_B
s_A	s_A	s_C	s_B	r_0	r_{240}	r_{120}
s_B	s_B	s_A	s_C	r_{120}	r_0	r_{240}
s_C	s_C	s_B	s_A	r_{240}	r_{120}	r_0

This is the multiplication table for triangle symmetries.

Abstract groups

An abstract group is a multiplication table: a set G with a product $\circ$ taking $g, h \in G$ and giving $g \circ h \in G$.

Product $\circ$ is required to have some properties (that are automatic for composition of symmetries. . .)

1. ASSOCIATIVITY: $g \circ (h \circ k) = (g \circ h) \circ k \quad (g, h, k \in G)$;
2. there's an IDENTITY $e \in G$: $e \circ g = g \quad (g \in G)$;
3. each $g \in G$ has INVERSE $g^{-1} \in G$, $g^{-1} \circ g = e$.

For symmetries, these properties are always true:

1. first doing (k then h), then doing g, is the same as first doing k, then doing (h then g);
2. doing g then doing nothing is the same as just doing g;
3. undoing a symmetry (putting things back where you found them) is also a symmetry.

Here's an example of a group with just two elements e and s. In fact it's the *only* example.

$\circ$	e	s
e	e	s
s	s	e

Approaching symmetry

Normal person's approach to symmetry:

1. look at something interesting;
2. find the symmetries.

This approach $\rightsquigarrow$ standard model in physics.

Explains everything that you can see without LIGO.

Mathematician's approach to symmetry:

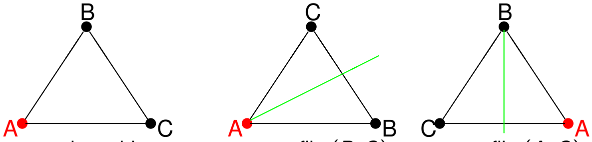
1. find all multiplication tables for abstract groups;
2. pick an interesting abstract group;
3. find something it's the symmetry group of;
4. decide that something must be interesting.

This approach $\rightsquigarrow$ Conway group (which has 8,315,553,613,086,720,000 elements) and Leech lattice (critical for packing 24-dimensional cannonballs).

Anyway, I'm a mathematician. . .

Which symmetries are really different?

Here are some of the symmetries of a triangle:



$e =$ do nothing $\quad s_A =$ flip (B, C) $\quad s_B =$ flip (A, C)

s_A and s_B are "same thing" from different points of view.

Can accomplish s_B in three steps: $\quad (A, B, C)$

1. flip (A, B) (apply s_C); $\quad (B, A, C)$
2. flip (B, C) (apply s_A); $\quad (C, A, B)$
3. unflip (A, B) (apply s_C^{-1}). $\quad (C, B, A)$

Summary: $s_B = s_C^{-1} s_A s_C$.

Defn. g, h conjugate if there's $k \in G$ so $h = k^{-1} g k$.

Three conjugacy classes of symmetries of triangle:

three reflections s_A, s_B, s_C (exchange two vertices);
two rotations r_{120}, r_{240} (cyclically permute vertices);
one trivial symmetry r_0 (do nothing).

Conjugacy classes

G any group; elements g and h in G are conjugate if there's k in G so $h = k^{-1}gk$.

Conjugacy class in G is an equivalence class.

$G =$ disjoint union of conjugacy classes

Example: $\underbrace{\triangle \text{ symms}}_{6 \text{ elts}} = \underbrace{\{\text{refls}\}}_{3 \text{ elts}} \cup \underbrace{\{\text{rotns}\}}_{2 \text{ elts}} \cup \underbrace{\{\text{identity}\}}_{1 \text{ elt}}$.

$6 = 3 + 2 + 1$ is class eqn for triangle symms.

G is abelian if $gh = hg \quad (g, h \in G)$.

G is abelian $\leftrightsquigarrow$ each conjugacy class is one element.

Size of conjugacy classes $\leftrightsquigarrow$ how non-abelian G is.

Conjugacy classes in S_n

$S_n =_{\text{def}}$ all ($n!$) rearrangements of $\{1, 2, \ldots, n\}$.

$=$ Symmetries of $(n-1)$-simplex: join n equidistant pts.

$S_3 = 6$ symms of triangle; $S_4 = 24$ symms of reg tetrahedron.

Typical rearrangement for $n = 5$: $g = \begin{pmatrix} 1\,2\,3\,4\,5 \\ 3\,5\,4\,1\,2 \end{pmatrix}$.

What g does: $(1 \to 3 \to 4 \to 1)(2 \to 5 \to 2)$.

Shorthand: $g = (134)(25)$: cycle (134) and (25)

This g is conjugate to $h = (125)(34) = \begin{pmatrix} 1\,2\,3\,4\,5 \\ 2\,5\,4\,3\,1 \end{pmatrix}$.

Theorem Any elt of S_n is a product of disjt cycles of sizes $p_1 \geq p_2 \geq \cdots \geq p_r$, $\sum p_j = n$. Two elts are conjugate $\leftrightsquigarrow$ have same cycle sizes.

Definition Partition of n is $p_1 \geq p_2 \geq \cdots \geq p_r$, $\sum p_j = n$.

Corollary Conj classes in S_n $\leftrightsquigarrow$ partitions of n.

Gelfand program...

...for using groups to do other math.

Say G is a group of symmetries of X.

Step 1: LINEARIZE. $X \rightsquigarrow V(X)$ vec space of fns on X. Now G acts by linear maps.

Step 2: DIAGONALIZE. Decompose $V(X)$ into minimal G-invariant subspaces.

Step 3: REPRESENTATION THEORY. Understand all ways that G can act by linear maps.

Step 4: PRETENDING TO BE SMART. Use understanding of $V(X)$ to answer questions about X.

One hard step is 3: how can G act by linear maps?

Definition of representation

G group; representation of G is

1. (complex) vector space V, and
2. collection of linear maps $\{\pi(g) \colon V \to V \mid g \in G\}$

subject to $\pi(g)\pi(h) = \pi(gh), \quad \pi(e) = \text{identity}$.

Subrepresentation is subspace $W \subset V$ such that

$$\pi(g)W = W \qquad (\text{all } g \in G).$$

Rep is irreducible if only subreps are $\{0\} \neq V$.

Irreducible subrepresentations are minimal nonzero subspaces of V preserved by all $\pi(g)$.

This is a group-theory version of eigenspaces.

There's a theorem like eigenspace decomp. . .

Diagonalizing groups

Theorem Suppose G is a finite group.

1. There are finitely many irr reps $\tau_1, \ldots, \tau_\ell$ of G.
2. Number ℓ of irr reps = number of conj classes in G.
3. Any rep π of G is sum of copies of irr reps:
$$\pi = n_1(\pi)\tau_1 + n_2(\pi)\tau_2 + \cdots + n_\ell(\pi)\tau_\ell.$$
4. Nonnegative integers $n_j(\pi)$ uniquely determined by π.
5. $|G| = (\dim \tau_1)^2 + \cdots + (\dim \tau_\ell)^2$.
6. G is abelian if and only if $\dim \tau_j = 1$, all j.

Dims of irr reps $\leftrightsquigarrow$ how non-abelian G is.

Two formulas for $|G|$:

$$\sum_{\text{conj classes}} \text{size of conj class} = |G| = \sum_{\text{irr reps } \tau} (\dim \tau)^2.$$

Same # terms each side; so try to match them up...

Partitions, conjugacy classes, representations

Recall S_n = perms of $\{1, \ldots, n\}$ symmetric group.

Recall $\pi = (p_1, \ldots, p_r)$ decr, $\sum p_i = n$ partition of n.

Partition $\leftrightsquigarrow$ array of boxes:  $\leftrightsquigarrow 9 = 4 + 3 + 1 + 1$.

Recall conjugacy class $C_\pi \leftrightsquigarrow$ partition π

Columns of π = cycle sizes of C_π.

Theorem. There is another bijection

$$(\text{irr representations of } S_n) \leftrightsquigarrow (\text{partitions of } n)$$

S_2			S_3			S_4		
$\lvert C_\pi \rvert$	π	$(\dim \tau_\pi)^2$	$\lvert C_\pi \rvert$	π	$(\dim \tau_\pi)^2$	$\lvert C_\pi \rvert$	π	$(\dim \tau_\pi)^2$
1		1	1		1	1		1
1		1	3		4	6		9
			2		1	3		4
						8		9
						6		1

Lessons learned

Conclusion: For S_n there is a natural bijection

$$\text{(conj classes)} \leftrightsquigarrow \text{(irr repns)}, \qquad C_\pi \leftrightsquigarrow \tau_\pi.$$

But $|C_\pi|$ not very close to $(\dim \tau_\pi)^2$.

This is math.

If what you want isn't true, change the universe.

Conjugacy classes in $GL(V)$: examples

V n-dimensional vector space over field F

Symms of V = rearrs of V resp $+$, scalar mult...

$\ldots =$ (invertible) linear transformations $= GL(V)$.

After choice of basis, these are invertible $n \times n$ matrices.

Say g and h are similar if there's invertible k so $h = k^{-1}gk$.

Means: g and h are "the same" up to change of basis.

{Similarity classes of matrices} = {conj classes in $GL(V)$}.

Examples for $n = 2$, $F = \mathbb{C}$ or $\mathbb{R}$:

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \qquad \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix} \qquad \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix} \qquad (\lambda_1, \lambda_2 \in F^\times)$$

Additional examples for $n = 2$, $F = \mathbb{R}$:

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \qquad (a + bi \in \overline{\mathbb{R}}, b \neq 0)$$

Conjugacy classes in $GL(V)$: general theory

If $F = \overline{F}$, conj class $\approx$ set of n eigvals in $F^\times = F - \{0\}$.

Better: conj class $\approx \underbrace{\text{multi-set of size } n \text{ in } F^\times}_{\text{count multiplicities}}$

Best: conj class $=$ function $\pi\colon F^\times \to$ partitions, $\sum_\lambda |\pi(\lambda)| = n$

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \leftrightsquigarrow \pi(\lambda_1) = \square, \quad \pi(\lambda_2) = \square \qquad \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix} \leftrightsquigarrow \pi(\lambda_1) = \square\square$$

$F \neq \overline{F}$: conj class $= \pi\colon$ Galois orbits $\Lambda \subset \overline{F}^\times \to$ partitions, $\sum_\Lambda |\pi(\Lambda)||\Lambda| = n$

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \leftrightsquigarrow \pi(\{a+bi, a-bi\}) = \square \qquad (b \neq 0)$$

$$\begin{pmatrix} a & -b & 1 & 0 \\ b & a & 0 & 1 \\ 0 & 0 & a & -b \\ 0 & 0 & b & a \end{pmatrix} \leftrightsquigarrow \pi(\{a+bi, a-bi\}) = \square\square \qquad (b \neq 0)$$

Conjugacy classes in $GL_n(\mathbb{F}_q)$

Seek (conj classes) $\overset{?}{\longleftrightarrow}$ (irr reps) for other groups.

Try next $GL_n(\mathbb{F}_q)$, invertible $n \times n$ matrices $/\mathbb{F}_q$.

$$\begin{aligned}|GL_n(\mathbb{F}_q)| &= (q^n-1)(q^n-q)\cdots(q^n-q^{n-1})\\ &= (q^{n-1}-1)(q^{n-2}-1)\cdots(q-1)\cdot 1\cdot q\cdots q^{n-1}\\ &= \underbrace{(1+q+\cdots+q^{n-1})(1+q+\cdots+q^{n-2})\cdots(1)}_{q\text{-analog of } n!}\\ &\quad \cdot (q-1)^n \cdot q^{n(n-1)/2}\end{aligned}$$

$GL_n(\mathbb{F}_q)$ is q-analogue of S_n.

Conj class in $GL_n(\mathbb{F}_q)$ = partition-valued function π on Galois orbits $\Lambda \subset \overline{\mathbb{F}_q}^\times$ such that

$$\sum_\Lambda |\underbrace{\Lambda}_{\text{eigval}}| \cdot \underbrace{|\pi(\Lambda)|}_{\text{mult}(\Lambda)} = n.$$

($\pi(\Lambda) = \emptyset$ for most Λ.)

What's that mean for $GL_2(\mathbb{F}_q)$?

Suppose q is an odd prime power.

Fix non-square $d \in \mathbb{F}_q$; $\mathbb{F}_{q^2} = \{a + b\sqrt{d} \mid a, b \in \mathbb{F}_q\}$.

Here are the conjugacy classes in $GL_2(\mathbb{F}_q)$:

1. Diagonalizable, two eigvals: $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ $(\lambda_1 \neq \lambda_2 \in \mathbb{F}_q^\times)$
2. Nondiagonalizable, two eigvals: $\begin{pmatrix} a & bd \\ b & a \end{pmatrix}$ $(0 \neq b \in \mathbb{F}_q)$
3. Nondiagonalizable, one eigval: $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ $(\lambda \in \mathbb{F}_q^\times)$
4. Diagonalizable, one eigval: $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ $(\lambda \in \mathbb{F}_q^\times)$.

$GL_n(\mathbb{F}_q)$: representations

Saw that conj class in $GL_n(\mathbb{F}_q)$ is partition-valued function on Galois orbits on $\bigcup_{d\geq 1} \mathbb{F}_{q^d}^\times$ (19th century linear algebra).

Similarly irr rep of $GL_n(\mathbb{F}_q)$ is partition-valued function on Galois orbits on $\bigcup_{d\geq 1} \widehat{\mathbb{F}_{q^d}^\times}$ (Green 1955).

$GL_2(\mathbb{F}_q)$

conj class C	# classes	$\lvert C\rvert$
diag, 2 ev	$(q-1)(q-2)/2$	$q(q+1)$
nondiag, 2 ev	$q(q-1)/2$	$q(q-1)$
$\square\square$	$q-1$	$(q+1)(q-1)$
$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	$q-1$	1
repn V	**# repns**	**$\dim \tau$**
princ series	$(q-1)(q-2)/2$	$q+1$
disc series	$q(q-1)/2$	$q-1$
$\square\square$	$q-1$	q
$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	$q-1$	1

Conclude (conj classes) $\leftrightarrow$ (irr reps).

Bijection has $\lvert C_\pi\rvert \approx (\dim V_\pi)^2$.

The rest of mathematics in one slide

There are similar ideas and questions for infinite groups.

Typical example is $GL(n, \mathbb{R})$, invertible real matrices.

Finding conjugacy classes is fairly easy.

Finding irreducible representations is harder (unsolved).

Finite group question

$$\text{dim(representation)} \overset{??}{\leftrightsquigarrow} \text{(size)}^{1/2}\text{(conjugacy class)}$$

becomes Lie group problem given conjugacy class C (a manifold) find a manifold X that's a "square root" of C:

$$C \overset{??}{\simeq} X \times X.$$

Same problem shows up in quantum mechanics.

So there's a reason to stay friendly with physicists.