Quaternionic groups

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The point of these notes is to relate some of the quaternionic groups discussed in class to complex matrix groups. In particular, the notation $Sp(n)$ for the quaternionic unitary group suggests that it has something to do with symplectic groups. This is true, and will be explained in (0.4b). There is some disagreement about whether symplectic groups should be labelled with $2n$ (the dimension of the vector space) or n (the "rank," which we'll talk about eventually). I think that for the noncompact groups $2n$ wins, but for the compact groups, there is more support for n. Anyway the result is that (0.4b) looks a bit funny, but that's life.

Recall the division ring of quaternions

$$\mathbb{H} = \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{R}\}. \tag{0.1a}$$

Multiplication is defined by

$$i^2 = j^2 = k^2 = -1, \qquad ij = k = -ji, \quad jk = i = -kj, \quad ki = j = -ik. \tag{0.1b}$$

A more succinct way to write this is

$$\mathbb{H} = \mathbb{C}[j], \qquad j^2 = -1, \quad jzj^{-1} = \overline{z}. \tag{0.1c}$$

That is, any quaternion h may be written uniquely as

$$h = z + jw \qquad (z, w \in \mathbb{C}); \tag{0.1d}$$

the multiplication rules are determined by (0.1c). There is an algebra anti-automorphism of the quaternions given by

$$\overline{a + bi + cj + dk} = a - bi - cj - dk, \qquad \overline{h_1 h_2} = \overline{h_2 h_1}. \tag{0.1e}$$

A right quaternionic vector space V is automatically a complex vector space, just by restricting scalar multiplication to the subring $\mathbb{C} \subset \mathbb{H}$. An

$\mathbb{H}$-linear map from V to V is automatically $\mathbb{C}$-linear, so we get an inclusion of $\mathbb{R}$-algebras

$$\operatorname{Hom}_{\mathbb{H}}(V,V) \hookrightarrow \operatorname{Hom}_{\mathbb{C}}(V,V), \tag{0.2a}$$

and an inclusion of Lie groups

$$GL_{\mathbb{H}}(V) \hookrightarrow GL_{\mathbb{C}}(V). \tag{0.2b}$$

$GL_{\mathbb{H}}(V)$ is *not* a complex Lie group, even though it has real dimension divisible by 2 (even 4!).

What tells you that the complex vector space V comes by "restriction of scalars" from a quaternionic vector space is the real-linear map

$$j\colon V \to V, \quad j(v\cdot z) = j(v)\cdot \overline{z}, \quad j^2 = -I. \tag{0.2c}$$

(By the second requirement, $j^{-1} = -j$.) We can define

$$\sigma_{\mathbb{H}}\colon \operatorname{Hom}_{\mathbb{C}}(V,V) \to \operatorname{Hom}_{\mathbb{C}}(V,V), \qquad \sigma_{\mathbb{H}}(T) = j^{-1}Tj = -jTj. \tag{0.2d}$$

Because of the two occurrences of j, $\sigma_{\mathbb{H}}(T)$ is complex-linear, so $\sigma_{\mathbb{H}}$ is well-defined; but the map $\sigma_{\mathbb{H}}$ itself is $\mathbb{C}$-conjugate linear. Now it is more or less obvious that

$$\operatorname{Hom}_{\mathbb{H}}(V,V) = \operatorname{Hom}_{\mathbb{C}}(V,V)^{\sigma_{\mathbb{H}}}, \qquad GL_{\mathbb{H}}(V) = GL_{\mathbb{C}}(V)^{\sigma_{\mathbb{H}}}. \tag{0.2e}$$

Now let's see how all this looks on the level of the compact subgroups. We start therefore with a positive definite Hermitian form $\langle \cdot,\cdot\rangle$ on V. Recall that this means (in addition to bi-additivity) that

$$\langle v\cdot h, w\rangle = \overline{h}\langle v,w\rangle, \qquad \langle v, w\cdot h'\rangle = \langle v,w\rangle h' \tag{0.3a}$$

and that

$$\langle v,w\rangle = \overline{\langle w,v\rangle}. \tag{0.3b}$$

(This is just like the definition of a Hermitian form on a complex vector space, except that I've put the conjugate-linearity in the first variable instead of the second. The reason is to simplify a lot of matrix formulas (in which the noncommutativity of $\mathbb{H}$ matters). Some mathematicians think that one should do the same thing with complex Hermitian forms, but I think that goes against too much tradition.) As long as V is finite-dimensional, any $\mathbb{H}$-linear transformation H of V has an *adjoint* H^* defined by the requirement

$$\langle v, Hw\rangle = \langle H^* v, w\rangle. \tag{0.3c}$$

The $\mathbb{R}$-linear algebra antiautomorphism $*$ of $\mathrm{Hom}_{\mathbb{H}}(V,V)$ gives rise to a group automorphism of order two

$$\theta_{\mathbb{H}}(g) = (g^*)^{-1} \qquad (g \in GL_{\mathbb{H}}(V)). \tag{0.3d}$$

The group of fixed points of this involution is (very easily seen to be) the quaternionic unitary group (of $\mathbb{H}$-linear transformations preserving the Hermitian form):

$$U_{\mathbb{H}}(V) = \{g \in GL_{\mathbb{H}}(V) \mid \theta_{\mathbb{H}}(g) = g\}. \tag{0.3e}$$

This kind of formula works for $\mathbb{R}$ and $\mathbb{C}$ as well: for example, if W is a finite-dimensional real vector space with a positive-definite quadratic form, then the adjoint operation for $\mathbb{R}$ (which is transpose on the level of matrices in an orthonormal basis) satisfies

$$\theta_{\mathbb{R}}(g) = (g^*)^{-1}, \quad U_{\mathbb{R}}(W) = O(W) = \{g \in GL_{\mathbb{R}}(W) \mid \theta_{\mathbb{R}}(g) = g\}.$$

The positive definite Hermitian form on the quaternionic vector space V defines a positive definite Hermitian form on the underlying complex vector space V, by the formula

$$\langle v, w\rangle_{\mathbb{C}} = a + bi \quad \text{whenever} \quad \langle v, w\rangle_{\mathbb{H}} = a + bi + cj + dk. \tag{0.3f}$$

(Like the quaternionic form from which it came, this complex Hermitian form is conjugate-linear in the first variable and linear in the second. If you prefer a Hermitian form the way they're usually defined, just interchange the arguments, or take the complex conjugate.) If $H \in \mathrm{Hom}_{\mathbb{H}}(V,V)$ is identified by (0.2a) with a complex linear map on V, then it's clear from this definition that

$$\text{quaternionic adjoint of } H = \text{complex adjoint of } H \in \mathrm{Hom}_{\mathbb{C}}(V,V) \tag{0.3g}$$

At the same time, we can define a complex symplectic form on V by

$$\omega(v, w) = c - di \quad \text{whenever} \quad \langle v, w\rangle_{\mathbb{H}} = a + bi + cj + dk. \tag{0.3h}$$

Equivalently,

$$\omega(v, w) = \langle vj, w\rangle_{\mathbb{C}}. \tag{0.3i}$$

(There is very little chance I got the signs exactly right in these formulas.) Any complex-linear transformation T has a symplectic adjoint ${}^{\omega}T$ defined by

$$\omega(Tv, w) = \omega(v, ({}^{\omega}T)w). \tag{0.3j}$$

We can compute this using (0.3i): the result is

$$ {}^{\omega}T = \sigma_{\mathbb{H}}(T^*). \tag{0.3k}$$

Formally it is clear that ${}^{\omega}({}^{\omega}T) = T$, so the symplectic adjoint is an involutive algebra antiautomorphism. In particular, this implies that

$$\text{the involutions } \sigma_{\mathbb{H}} \text{ and } * \text{ commute with each other.} \tag{0.3l}$$

If we define an involutive automorphism by

$$\tau_{Sp}(g) = ({}^{\omega}g)^{-1} = \sigma_{\mathbb{H}} \circ \theta_{\mathbb{C}} \qquad (g \in GL_{\mathbb{C}}(V)) \tag{0.3m}$$

then it is clear that the complex symplectic group is the group of fixed points:

$$Sp(V,\omega) = GL_{\mathbb{C}}(V)^{\tau_{Sp}}. \tag{0.3n}$$

So here is what we know.

1. $GL_{\mathbb{H}}(V)$ is the group of fixed points of the involution $\sigma_{\mathbb{H}}$ acting on $GL_{\mathbb{C}}(V)$ (see (0.2e)).

2. The complex unitary group $U_{\mathbb{C}}(V)$ is the group of fixed points of the involution $\theta_{\mathbb{C}}$ acting on $GL_{\mathbb{C}}(V)$ (complex version of (0.3e)).

3. The quaternionic unitary group $U_{\mathbb{H}}(V)$ is the group of fixed points of the involution $\theta_{\mathbb{H}}$ acting on $GL_{\mathbb{H}}(V)$ (0.3e). This in turn is the same as the group of fixed points of $\theta_{\mathbb{C}}$ acting on $GL_{\mathbb{H}}(V) \subset GL_{\mathbb{C}}(V)$ (because of (0.3g)).

4. The involutions $\theta_{\mathbb{C}}$ and $\sigma_{\mathbb{H}}$ of $GL_{\mathbb{C}}(V)$ commute with each other (0.3l).

5. The complex symplectic group $Sp(V,\omega)$ is the group of fixed points of the involution $\tau_{Sp} = \sigma_{\mathbb{H}} \circ \theta_{\mathbb{C}}$ (0.3n) acting on $GL_{\mathbb{C}}(V)$.

Putting these facts together, we get

$$\begin{aligned} U_{\mathbb{H}}(V) &= \{g \in GL_{\mathbb{C}}(V) \mid \theta_{\mathbb{C}}(g) = g \text{ and } \sigma_{\mathbb{H}}(g) = g\} \\ &= GL_{\mathbb{H}}(V) \cap U_{\mathbb{C}}(V). \end{aligned} \tag{0.4a}$$

We can rewrite (0.4a) as

$$\begin{aligned} U_{\mathbb{H}}(V) &= \{g \in GL(2n,\mathbb{C}) \mid \theta_{\mathbb{C}}(g) = g \text{ and } \tau_{Sp}(g) = g\} \\ &= Sp(V,\omega) \cap U_{\mathbb{C}}(V) : \end{aligned} \tag{0.4b}$$

the quaternionic unitary group may be identified with the complex-linear transformations of V preserving both the complex symplectic form and the (positive) complex Hermitian form arising from the quaternionic Hermitian form.

If $V = \mathbb{H}^n$, then $U_{\mathbb{H}}(V) = Sp(n)$, and $Sp(V,\omega) = Sp(2n,\mathbb{C})$; so (0.4b) is the relationship we want between the quaternionic unitary group and the complex symplectic group.