18.100B Problem Set 9

Due in class Monday, May 4. You may discuss the problems with other students, but you should write solutions entirely on your own.

1. Find a sequence $\{f_n\}$ of non-negative continuous functions on $[0, 1]$ such that

$$\lim_{n\to\infty} f_n(x) = 0, \qquad \text{all } x \in [0, 1],$$

and also

$$\int_0^1 f_n(x)dx = 1.$$

(So for this sequence, you can't interchange integration and the limit.)

2. Suppose $\{f_n\}$ is a sequence of functions on $[0, 1]$ with the following properties:

a) f_n is continuous on $[0, 1]$;
b) f_n is differentiable on $(0, 1)$;
c) there is a constant M (independent of n) so that $|f_n'| \leq M$; and
d) f_n converges pointwise on $[0, 1]$ to a limit function f.

Prove that the sequence must converge uniformly (so that the limit function is necessarily continuous). (Hint: by the Mean Value Theorem, the functions f_n cannot change by more than hM on an interval of length h. So break $[0, 1]$ into a finite number of intervals each of length no more than (say) ϵ/M. By choosing n large enough, you can make f_n close to f at all the endpoints of your intervals (since there are only finitely many of those). Now deduce that f_n has to be close to f everywhere.)

3. (This problem is a version of the one in the text, page 169, number 21. The point of it is to show the need for one of the hypotheses in Theorem 7.33; but you don't need to know about that theorem to do the problem. For a hint, see the problem in the text.) Let

$$\mathcal{A} = \text{functions } f(t) = \sum_{n=0}^{N} c_n e^{int}, \text{ with } N \text{ a non-negative integer and } c_n \in \mathbb{C}.$$

This is an algebra of continuous complex-valued functions on $[0, 2\pi]$, all satisfying $f(0) = f(2\pi)$. Prove that there is a continuous complex-valued function g on $[0, 2\pi]$ such that $g(0) = g(2\pi)$, but nevertheless

$$\sup_{x\in[0,2\pi]} |g(x) - f(x)| \geq 1, \qquad f \in \mathcal{A}.$$

4. Text, page 170, number 24.