18.100B Problem Set 8 Partial Solutions

1. This is a version of problems 8 and 10 in the text (pages 166-167). Let $S = \{x_1, x_2, \dots\}$ be a countable subset of $(0,1)$. (For example, S might consist of all the rational numbers between 0 and 1.) Define a real-valued function f on $[0,1]$ by

$$f(x) = \sum_{\{n | x_n \leq x\}} 2^{-n}.$$

The notation means that the sum extends exactly over those positive integers n for which $x_n \leq x$.

a) Show that f is a well-defined increasing function on $[0,1]$, that $f(0) = 0$, and $f(1) = 1$. (By Theorem 6.9, it follows that f is Riemann-integrable.)

b) Show that f is continuous at x if and only if $x \notin S$.

c) Suppose that $x \notin \overline{S}$. Prove that f is differentiable at x.

d) Suppose S is dense in $[0,1]$. Does the derivative $f'(x)$ exist for any value of x? (In this case part (c) doesn't provide any places where the derivative exists. This question is quite a bit harder than any of the others; don't worry if you can't make any progress on it.)

The answer is that the derivative *does* exist (and is equal to zero) for a large and dense set of values of x. Here is an outline of a proof of that statement. The proof is a relative of the proof of Theorem 7.18 (that there is a nowhere differentiable continuous function), even though the conclusion is almost the opposite.

So we want to find values of x where the derivative of f is zero. This means that we want to be able to make $|f(t) - f(x)|$ much smaller than $|t - x|$ by making $|t - x|$ small enough (but not zero). I'll talk only about the case $t > x$; the other case is almost identical. First notice that

$$|f(t) - f(x)| = \sum_{\{n | x < x_n \leq t\}} 2^{-n}. \tag{1}$$

For each positive integer r, define

$$m(r,x) = \text{smallest value of } m \text{ with } x_m \in [x - 2^{-r}, x + 2^{r}]; \tag{2}$$

if the interval contains no element of S, then define $m(r,x) = +\infty$. The reason for making this definition is that (1) implies

$$f(t) - f(x) = \sum_{\{n | x < x_n \leq t\}} 2^{-n} \leq \sum_{n \geq m(r,x)} 2^{-n} = 2^{-m(r,x)+1} \quad (t \in [x, x + 2^{-r}]). \tag{3}$$

Any t that is both greater than x and less than or equal to 1 belongs to exactly one interval $(x + 2^{-r-1}, x + 2^{-r}]$, for a non-negative integer $r = r(t)$. Therefore $t - x \geq 2^{-r(t)-1}$, and

$$(f(t) - f(x))/(t - x) \leq 2^{-m(r,x)+1} \cdot 2^{r+1} = 2^{r - m(r,x)+2} \qquad (r = r(t)). \tag{4}$$

Taking the limit as t approaches x means making $r(t)$ go to infinity. If you think about this and (4) for a bit, you should see

Lemma A. *Fix* $x \in [0,1]$, *and define* $m(r,x)$ *by (2) above. If* $\lim_{r\to\infty} r - m(r,x) = -\infty$, *then* $f'(x) = 0$.

Probably the converse of this statement is also true.

So how can we find values of x for which $r - m(r,x)$ tends to $-\infty$? Any element $x = x_m$ of is excluded, for $m(r,x_m) \le m$, and so $\lim_{r\to\infty} r - m(r,x_m) = +\infty$. That's no surprise, since we already knew f was discontinuous at x_m. Here is one way to win. Fix a big positive integer N, and a small positive number ϵ.

Lemma B. *Suppose*

$$x \notin B(N,\epsilon) = \{x_1, \ldots, x_{N-1}\} \bigcup_{m=N}^{\infty} [x_m - 2^{-m/(1+\epsilon)}, x_m + 2^{-m/(1+\epsilon)}].$$

Then $f'(x) = 0$.

The notation B stands for "bad." The reason this lemma is true is that if $x \notin B(N,\epsilon)$, and r is so big that $2^{-r} > |x - x_m|$ for all $m < N$, then

$$m(r,x) \ge (1+\epsilon)r.$$

Consequently

$$\lim_{r\to\infty} r - m(r,x) \le \lim_{r\to\infty} r - (1+\epsilon)r = \lim_{r\to\infty} -\epsilon r = -\infty.$$

The last thing to prove is that there are points not in $B(N,\epsilon)$. I won't prove this, but the following lemma is supposed to be convincing evidence.

Lemma C. *The sum of the lengths of the intervals comprising* $B(N,\epsilon)$ *is at most* $2 \cdot 2^{-N/(1+\epsilon)}/(1 - 2^{-N/(1+\epsilon)})$. *Given any positive* ϵ, *this bound can be made as small as desired by choosing* N *large enough.*

It's intuitively reasonable that if the sum of the lengths of the intervals in $B(N,\epsilon)$ is less than 1, then $B(N,\epsilon)$ should not be all of $[0,1]$. I don't know an easy proof of this fact; it follows from the existence of Lebesgue measure (Chapter 11 of Rudin). What Lemma C shows is that $f'(x)$ exists and is equal to zero except for x in a set of measure 0.

2. Text, page 165, number 3: find uniformly convergent sequences f_n and g_n so that $f_n g_n$ is not uniformly convergent.

Write f and g for the limit functions. Then

$$|f_n g_n - fg| = |f_n g_n - f_n g + f_n g - fg| \le |f_n| \cdot |g_n - g| + |f_n - f| \cdot |g|.$$

From this it's clear that if f and g are bounded, then $f_n g_n$ is uniformly convergent. So we need a uniformly convergent sequence of unbounded functions. Lots of things work; one possibility is $E = (0,1)$,

$$f_n(x) = g_n(x) = (1/x) + (1/n).$$

These converge uniformly to $1/x$, so f_n^2 converges pointwise to $f^2 = 1/x^2$; but

$$|f_n^2(x) - f^2(x)| = 2/(nx) + 1/n^2 > 2/nx,$$

which is unbounded on $(0,1)$; so the convergence is not uniform.