18.700 Problem Set 2

Due in class Friday October 6; late work will not be accepted. Your work on graded problem sets should be entirely your own. Unless otherwise noted, the field of scalars is the real numbers.

1. (10 points) In each part, find all left and right inverses of the indicated matrices, or explain why they do not exist.

a) $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. (Here θ is a real number. Be careful about dividing by zero.)

b) $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \end{pmatrix}$.

c) The problem of finding polynomials that take specified values at specified points is called *interpolation*. The inverse you found in part (b) says that a certain interpolation problem always has a solution, and tells how to find one. Explain.

2. (10 points) Suppose that A is an $m \times n$ matrix. This problem essentially gives an algorithm to find a basis of the null space of A^t. Enlarge A to an $m \times (n+m)$ matrix $A' = (A|I_m)$. Perform elementary row operations on A' to put it in reduced echelon form $\widetilde{A'} = (\widetilde{A}, U)$; here $\widetilde{A}$ is the reduced echelon form of A, and U is an $m \times m$ matrix.

a) Explain why $UA = \widetilde{A}$, and why the rows of U are linearly independent.

b) Suppose that the rank of A is r. Explain why the last $m - r$ rows of $\widetilde{A}$ are zero.

c) Write $u_{r+1}, u_{r+2}, \ldots, u_m$ for the last $m - r$ rows of U; these are row vectors in $\mathbb{R}^m$. Explain why $u_i A = 0$ for $i = r+1, \ldots, m$.

3. (10 points) Suppose that $\mathbb{F}$ is any field. Consider the collection $\mathbb{E}$ of all symbols $a + b\sqrt{2}$, with a and b in $\mathbb{F}$. (The square root sign is not supposed to mean anything here; $\sqrt{2}$ is just a symbol, like the complex number i, that is defined to have certain properties.) There is an obvious way to define addition on these symbols:

$$(a + b\sqrt{2}) + (c + d\sqrt{2}) = (a + c) + (b + d)\sqrt{2}.$$

There is an almost obvious way to define multiplication:

$$(a + b\sqrt{2})(c + d\sqrt{2}) = (ac + 2bd) + (ad + bc)\sqrt{2}.$$

The book (more or less) says on page 85 that these definitions make $\mathbb{E}$ into a field when $\mathbb{F} = \mathbb{Q}$, the field of rational numbers. For other fields $\mathbb{F}$, things can be trickier. It turns out that conditions (i)–(iii) in Definition 1.1.1 in the book are always satisfied in $\mathbb{E}$; the proofs are easy, and you can just assume that.

a) If $\mathbb{F} = \mathbb{R}$, the field of real numbers, explain why $\mathbb{E}$ is *not* a field.

b) If $\mathbb{F} = \{1, 2, 3\}$ is the field of integers modulo 3 (see pages 132–133 in the text), prove that $\mathbb{E}$ is a field consisting of exactly nine elements.

c) What happens if $\mathbb{F}$ is the field of integers modulo 7?