1. Homework 9

Problem. The Weyl algebra is the algebra of differential operators on $\mathbb{R}^n$ with polynomial coeffiecients. Consider the subalgebra $\mathfrak{g}$ of the Weyl algebra that's generated by differential opperators $\frac{\partial^2}{\partial x_i \partial x_j}$ and $x_i x_j$ for $1 \leqslant i \leqslant j \leqslant n$.

(1) Find a basis of $\mathfrak{g}$;

(2) Find a Cartan subalgebra $\mathfrak{h}$ and the root space decomposition of $\mathfrak{g}$;

(3) Choose a set of positive roots $R^+(\mathfrak{g}, \mathfrak{h})$ so that the nilradical $\mathfrak{n}$ contains $\frac{\partial^2}{\partial x_i \partial x_j}$ for all $1 \leqslant i \leqslant j \leqslant n$;

(4) The Weyl algebra acts on $V = \mathbb{C}[x_1, x_2, \ldots, x_n]$. When restricted to $\mathfrak{g}$, V is an infinite dimensional representation of $\mathfrak{g}$. Prove that V is a direct sum of two irreducible highest weight modules of $\mathfrak{g}$.

2. Homework 10

Problem. Suppose that G is a compact Lie group and acts smoothly on a locally compact space X and thus on $C(X)$. Suppose (π_1, E_1) and (π_2, E_2) are two irreducible representations and $\phi_i : E_i \to C(X)$ are injections respecting the G-action. Define $\phi_{E_1 \otimes E_2} \to C(X)$ by $e_1 \otimes e_2 \mapsto \phi_1(e_1)\phi_2(e_2)$ using the universal property of the tensor product.

(1) Show that $\phi_{E_1 \otimes E_2}$ respects the G-action.

(2) Given any E_1, E_2, show that we can find X and $\phi_i (i = 1, 2)$, such that $\phi_{E_1 \otimes E_2}$ is injective.

(3) Prove the equivalence of the following three statements:

(a) (π, E) appears in $C(X)$

(b) There exists $x \in X$, such that (π, E) appears in $C(G \cdot x)$.

(c) $\pi|_{G_x}$ contains the trivial representation of G_x (the isotropy group of G fixing x).