1. Homework 1

Problem 1. Write 2 or 3 sentences to explain why

$$\dim O(n) = (n-1) + (n-2) + \cdots + 1 = n(n-1)/2.$$

Problem 2. Prove: if f is a polynomial in n variables and $fr^2 - f$ is homogenous of degree m, then $f = 0$. (Here $r^2 = x_1^2 + \cdots + x_n^2$.)

Problem 3. Find an $O(2)$ invariant subspace H^m of the space of homogenous polynomials of degree m in $z, \bar{z}$ with the property that $V^{[m]} \cong V^{[m-2]} \oplus H^m$.

2. Homework 2

Problem 1. Give a formula for the dimension of

$$H^m = \{f \mid f \text{polys in } n \text{ vars homog of deg } m \text{ with } \Delta(f) = 0\}.$$

Problem 2. Prove: $S^m((\mathbb{R}^n)^*) = H^m \oplus r^2 H^{m-2} \oplus r^4 H^{m-4} \oplus \cdots \oplus r^{2[m/2]} H^{m-2[m/2]}$.

3. Homework 3

Problem 6. Define $H = \ell^2$, the set of sequences $\{(a_1, a_2, a_3, \dots) \mid a_i \in \mathbb{C} \text{ and } \sum |a_i|^2 < \infty\}$. Fix $\lambda_1, \lambda_2, \lambda_3, \dots$ a sequence of real numbers and define: $T(a_1, a_2, a_3, \dots) = (\lambda_1 a_1, \lambda_2 a_2, \lambda_3 a_3, \dots)$. Then

- T is well defined $\iff \lambda_i$ is bounded.
- T is always self-adjoint.

Show that the operator T is compact $\iff$ the sequence $\{\lambda_i\} \to 0$.

4. Homework 4

Problem 1. Let $G = SU(2) \cong S^3$. Then $G \times G$ acts on $L^2(S^3)$. Decompose $L^2(S^3) = \bigoplus_\pi V_\pi^* \times V_\pi$, where π is an irreducible representation of $SU(2)$. Relate this to the decomposition of $L^2(S^3) = H^1 \oplus H^2 \oplus H^3 \dots$ under $O(4)$.

Problem 2. Let $G = O(3)$. Define $(V_\pi, \pi) = H^m =$ harmonic polynomials of deg $m \cong S^m / r^2 S^{m-2}$. Consider $O(2) \subset O(3)$ with

$$O(2) = \left\{ \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix} = \text{ orthogonal group in } x, y \text{ variables} \right\}.$$

(a) Show that any $O(2)-$ invariant polynomial in x, y, z is a polynomial in variables $r_2^2 = x^2 + y^2$ and z.

(b) Deduce $\dim(H^m)^{O(2)} = 1$. Here $(H^m)^{O(2)}$ is the subspace of $O(2)-$invariant vectors in H^m.

(c) Deduce that H^m is an irreducible representation of $O(3)$.

5. Homework 5

Problem 1. Prove that $\pi_{3111} \ncong \pi_{222}$. Here if $\mu = (m_1, m_2, \ldots, m_r)$ is a partition of n, then π_μ is the unique irreducible representation of S_n that corresponds to the partition μ.

Problem 2. $G = SO(2n+1) = (2n+1) \times (2n+1)$ real orthogonal matrices. Find a maximal torus inside G and prove that your answer is correct.

6. Homework 6

Problem 1. Let $G = SO(3)$ and $T = SO(2)$, which is the upper left corner of G. Then $\mathfrak{g}_{\mathbb{C}} \cong$ skew symmetric 3×3 complex matrices. Write down $\mathfrak{t}, \mathfrak{g}_\alpha^{\mathbb{C}}, \mathfrak{g}_{-\alpha}^{\mathbb{C}}, \mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}$ and prove that $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$.

Problem 2. G is any group. Show that each irreducible, finite-dimensional real representation of G is obtained in one of the following ways.

- $(\pi, V_{\mathbb{C}})$ is an irreducible complex representation of G with the property that $\tilde{\pi} \neq \pi$.
  Then form: $W_{\mathbb{C}} = V_{\mathbb{C}} \oplus \bar{V}_{\mathbb{C}}$ and complex representation $\pi \otimes \bar{\pi}$. Define a complex conjugation on $W_{\mathbb{C}}$ by $\sigma(u, v) = (v, u)$. In this case
  $$\dim(\text{real repn}) = 2 \dim(\text{complex repn}).$$
- $(\pi, V_{\mathbb{C}})$ is an irreducible representation of G s.t. $\tilde{\pi} \cong \pi$. Choose an intertwining operator $T : V_{\mathbb{C}} \to \bar{V}_{\mathbb{C}}$. Show that one can choose T such that $T^2 = +1$ or -1.
  - If $T^2 = 1$, then $W_{\mathbb{C}} = V_{\mathbb{C}}$, $\sigma = T$ and the real representation has the same dimension as the complex representation.
  - If $T^2 = -1$, explain how to find σ.

Problem 3. Give an example of the last possibility above.

7. Problem Set 7

Problem 1. Suppose A is $n \times n$ matrix over $\mathbb{C}$. Prove the following are equivalent:

- $\exp(2\pi i A) = \mathrm{Id}$;

- A is diagonalizable, with eigenvalues in $\mathbb{Z}$.

Problem 2. Given the homomorphism

$$\phi_\alpha \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{pmatrix},$$

describe its extension to a homomorphism $\phi_\alpha : SU(2) \to O(4)$.

Problem 3. Prove $Z(G) = \{h \in H \mid \alpha(h) = 1, \forall \alpha \in R(G,H)\}$.

Problem 4. H is any torus. X_* = lattice of 1-parameter subgroups, X^* = lattice of characters. Suppose $R \subset X^*$ is a finite set. Define

$L = \mathbb{Z}R$ sublattice of X^* that's generated by R.

$\bar{L} = \{\lambda \in X^* \mid N\lambda \in L,$ for some large $N\}$, a sublattice of X^*.

$Z = \{h \in H \mid \alpha(h) = 1, \forall \alpha \in R\}$.

Show that $\bar{L}/L$ is a finite abelian group and is isomophic to the group of characters of finite abelian group Z/Z_0, where Z_0 is the identity component of Z.

In particular, the number of connected components of Z is equal to the index of L in $\bar{L}$.

8. Homework 8

Problem 1. Prove that $U(n) = \{$the group of unitary matrices$\}$ is homeomorphic to $U(1) \times SU(n)$, but that $U(n)$ is not isomorphic to $U(1) \times SU(n)$ as compact Lie groups.