Notes on Picard's method

In class on Friday I was sloppy about a number of things; so here is a slightly more careful account of the existence of solutions to systems of differential equations with a Lipschitz condition.

A *box* $B \subset \mathbb{R}^n$ is

$$B = \{y = (y_1, \ldots, y_n) \in \mathbb{R}^n \mid a_1 \leq y_1 \leq b_1, \ldots a_n \leq y_n \leq b_n\}. \qquad (BOX)$$

Here $a_1 < b_1, a_2 < b_2, \ldots a_n < b_n$ are real numbers. A short way to write these conditions is $y_i \in [a_i, b_i]$. The *interior of the box* consists of y such that $a_i < y_i < b_i$ for all i; we write $y_i \in (a_i, b_i)$.

Suppose $F\colon B \to \mathbb{R}^m$ is a function defined on B and taking values in $\mathbb{R}^m$. Recall that F is said to *satisfy a Lipschitz condition* if there is a constant L so that

$$|F(y) - F(y')| \leq L|y - y'| \qquad (y, y' \in B). \qquad (LIP)$$

(The absolute value signs all indicate lengths of vectors, in $\mathbb{R}^m$ on the left and $\mathbb{R}^n$ on the right.)

A fundamental (and not obvious) property of continuous functions on boxes is that they are bounded: if F is continuous on B, then there is a constant M so that

$$|F(y)| \leq M \qquad (y \in B). \qquad (BDD)$$

Before beginning the existence theorem, let me remark that the Lipschitz condition is a very strong version of what is *uniform continuity*. Here is what it means for F to be continuous: for every $b \in B$, and every $\epsilon > 0$, there is a constant $\delta > 0$ so that $|y - b| \leq \delta$ implies that $|F(y) - F(b)| \leq \epsilon$. To emphasize that the constant δ depends both on b and on ϵ, I'll write it as $\delta(b, \epsilon)$. Then the continuity condition at b reads

$$|y - b| \leq \delta(b, \epsilon) \Rightarrow |F(y) - F(b)| \leq \epsilon \qquad (y \in B). \qquad (CONT)$$

Here is what it means for F to be uniformly continuous: for every $\epsilon > 0$ there is a constant $\delta(\epsilon) > 0$ so that

$$|y - b| \leq \delta(\epsilon) \Rightarrow |F(y) - F(b)| \leq \epsilon \qquad (y, b \in B). \qquad (UNIF)$$

That is, the constants $\delta(b, \epsilon)$ can be chosen independent of b—"uniformly" in B. Now the remark is that if F satisfies the Lipschitz condition (LIP) above, then it must also satisfy the uniform continuity condition $(UNIF)$. Specifically, you can choose $\delta(\epsilon) = \epsilon/L$ (which is independent of b, as required).

So here is the existence theorem.

Theorem. *Assume that $F(y, t)$ is a continuous function defined on a box B in $\mathbb{R}^{n+1}$, taking values in $\mathbb{R}^n$; and assume that F satisfies the Lipschitz condition (LIP) in the y variables, and that F is bounded by M on the box (see (BDD)).*

Assume that (y_0, t_0) is a point in the interior of B. Write r for the distance from y_0 to the boundary of B. Fix any real number $t_1 > t_0$ so that (y_1, t_1) is still in B, and also

$$e^{L(t_1 - t_0)} \leq rL/M.$$

Then the initial value problem

$$y' = F(t, y), \qquad y(t_0) = y_0 \tag{IVP}$$

has a solution defined on the interval $[t_0, t_1]$.

Proof. Just as in the case of one variable, the equation (IVP) is equivalent to the integral equation

$$y(t) = y(t_0) + \int_{t_0}^{t} F(y(s), s)ds \qquad (t \in [t_0, t_1]) \tag{$INTEG$}$$

It is this integral equation that I'll actually solve. The idea is to construct a sequence of successive approximations $y_0(t), y_1(t), \ldots$, and to define $y(t)$ as the limit of these approximations.

The initial approximation is the constant function

$$y_0(t) = y_0, \qquad t \in [t_0, t_1].$$

The later approximations are defined recursively by the formula

$$y_{m+1}(t) = y_0 + \int_{t_0}^{t} F(y_m(s), s)ds \qquad (m \geq 1, t \in [t_0, t_1]). \tag{PIC}$$

The reason for the complicated assumption on t_1 is to make sure that the successive approximations are all well-defined: that $y_m(s)$ takes values in B, so that $F(y_m(s), s)$ is defined. I'm not going to discuss that issue carefully; it's not serious.

We need to get some control on the difference between y_{m+1} and y_m. Here is the result.

Lemma. *In the setting above, we have for every $m \geq 0$*

$$|y_{m+1}(t) - y_m(t)| \leq ML^m(t - t_0)^{m+1}/(m+1)! \qquad (t \in [t_0, t_1]).$$

Proof. The proof is by induction on m. For $m = 0$, the definition (PIC) says that

$$y_1(t) - y_0(t) = \int_{t_0}^{t} F(y_0, s)ds.$$

Taking lengths and using the bound M on F gives

$$|y_1(t) - y_0(t)| \leq \int_{t_0}^{t} |F(y_0, s)|ds \leq \int_{t_0}^{t} Mds = M(t - t_0).$$

This is exactly the case $m = 0$ claimed in the lemma.

So suppose that $m > 0$, and that we know the estimate in the lemma for $m - 1$. Then the definition (PIC) gives

$$y_{m+1}(t) - y_m(t) = \int_{t_0}^{t} (F(y_m(s), s) - F(y_{m-1}(s), s))ds.$$

Taking lengths and applying the Lipschitz condition L gives

$$|y_{m+1}(t) - y_m(t)| \leq \int_{t_0}^{t} |F(y_m(s), s) - F(y_{m-1}(s), s)|ds \leq \int_{t_0}^{t} L|y_m(s) - y_{m-1}(s)|ds.$$

Applying the case $m - 1$ of the lemma gives

$$|y_{m+1}(t) - y_m(t)| \leq \int_{t_0}^{t} L \cdot ML^{m-1}(s - t_0)^m/m!ds.$$

Performing the integral gives

$$|y_{m+1}(t) - y_m(t)| \leq ML^m(t - t_0)^{m+1}/(m + 1)!,$$

which completes the inductive step. Q.E.D

End of proof of Theorem. The argument on page 682 of the text shows that the sequence of functions y_m converges uniformly to some limit y on the interval $[t_0, t_1]$. As the uniform limit of a sequence of continuous functions, y is continuous. We can evaluate the integral

$$y_0 + \int_{t_0}^{t} F(y(s), s)ds = y_0 + \int_{t_0}^{t} \lim_{m\to\infty} F(y_m(s), s)ds.$$

Since the convergence is uniform, we can take the limit outside the integral sign, getting

$$y_0 + \int_{t_0}^{t} F(y(s), s)ds = \lim_{m\to\infty} \left(y_0 + \int_{t_0}^{t} F(y_m(s), s)ds \right) = \lim_{m\to\infty} y_{m+1}(t) = y(t).$$

That is, y is a solution of the equation $(INTEG)$, as we wished to show. Q.E.D.