18.704: Seminar in Algebra and Number Theory
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Orthogonal Geometry
(char $F \neq 2$)

Let V be a quadratic space of dimension $n \geq 2$ over a field F, with char $F \neq 2$. Let B be a non-degenerate orthogonal form on V. The isometries of V (called orthogonal transformations) comprise the orthogonal group $O(V)$. Thus,

$$O(V) = \{\tau \in GL(V) \mid B(\tau u, \tau v) = B(u,v), \text{ all } u,v \in V\} \leq GL(V)$$

If $\tau \in V$ and $\{v_1, \ldots, v_n\}$ is a basis for V then $\tau \in O(V)$ if and only if $T^t\widehat{B}T = \widehat{B}$, where T is the matrix representing τ relative to the basis.

Proof. Case I ($\tau \in O(V) \Rightarrow T^t\widehat{B}T = \widehat{B}$): Suppose $\tau \in O(V)$. Then, we know that if $u, v \in V$, then $B(\tau u, \tau v) = B(u,v)$. By definition of $\widehat{B}$, we know that $B(u,v) = u^t\widehat{B}v$. Thus, $B(\tau u, \tau v) = (\tau u)^t\widehat{B}(\tau v) = u^tT^t\widehat{B}Tv$. And, since $B(\tau u, \tau v) = B(u,v)$, we know that

$$B(\tau u, \tau v) = u^tT^t\widehat{B}Tv = u^t\widehat{B}v = B(u,v)$$

We know that $u^tT^t\widehat{B}Tv = u^t\widehat{B}v$ is true for all vectors $u, v \in V$. We also know that $\widehat{B}_{ij}$, the ij-th entry of $\widehat{B}$, is equal to $e_i^t\widehat{B}_{ij}e_j$, where e_i is a basis vector with 1 in the i-th place and 0 elsewhere and e_j is a basis vector with 1 in the j-th place and 0 elsewhere. Then, $u^tT^t\widehat{B}Tv = u^t\widehat{B}v$ implies that $\left({}^tT^t\widehat{B}T\right)_{ij} = \widehat{B}_{ij}$ for $1 \leq i \leq n$ and $1 \leq j \leq n$. Therefore, since every ij-th entry of ${}^tT^t\widehat{B}T$ equals the ij-th entry of $\widehat{B}$ it follows that $T^t\widehat{B}T = \widehat{B}$.

Case II ($T^t\widehat{B}T = \widehat{B} \Rightarrow \tau \in O(V)$): Suppose $T^t\widehat{B}T = \widehat{B}$. Let $u, v \in V$. Then

$$\begin{aligned} T^t\widehat{B}T = \widehat{B} &\Rightarrow u^tT^t\widehat{B}T = u^t\widehat{B} \\ &\Rightarrow u^tT^t\widehat{B}Tv = u^t\widehat{B}v \\ &\Rightarrow (\tau u)^t\widehat{B}(\tau v) = u^t\widehat{B}v \\ &\Rightarrow B(\tau u, \tau v) = B(u,v) \end{aligned}$$

As a result,

$$\begin{aligned} T^t\widehat{B}T = \widehat{B} &\Rightarrow \det(T^t\widehat{B}T) = \det(\widehat{B}) \\ &\Rightarrow \det(T^t)\det(\widehat{B})\det(T) = \det(\widehat{B}) \\ &\Rightarrow \det(T)^2\det(\widehat{B}) = \det(\widehat{B}) \end{aligned}$$

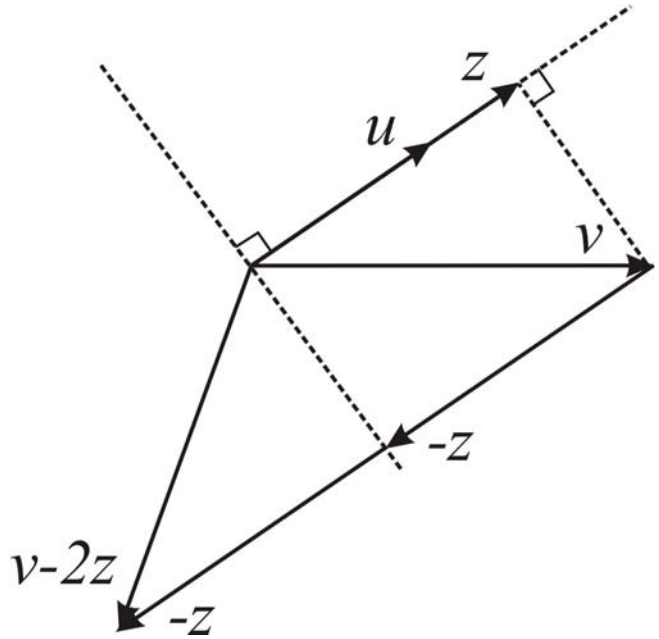
Since B is non-degenerate, we know that $\det(\widehat{B}) \neq 0$. Therefore, from the preceding expression, we see that $\det(T)^2 = 1$, which implies that $\det T = \pm 1$. Since char $F \neq 2$, $1 \neq -1$, so we have two cases: (1) $\det \tau = 1$, in which case τ is called a rotation (or a proper orthagonal transformation); and, (2) if $\det \tau = -1$, in which case τ is called a reversion (or an improper orthagonal transformation).

The rotations in $O(V)$ form the special orthogonal group $SO(V)$. The special orthogonal group is clearly nonempty because if we let $T = I_n$ be the matrix representing τ relative to the basis, then we have that $T^t\widehat{B}T = \widehat{B}$, which implies that $\tau \in O(V)$ (as we showed above). And, since $\det T = \det I_n = 1$, we know that $\tau \in SO(V)$. So, we know that proper orthogonal transformations exist in $O(V)$.

We now show that improper orthogonal transformations exist in $O(V)$. Let $u \in V$, be any vector such that $B(u,u) \neq 0$ (ie. u is anisotropic). We define a linear transformation σ_u to be:

$$\sigma_u(v) = v - 2\frac{B(v,u)}{B(u,u)}u$$

for all $v \in V$. Defined in this way, σ_u is the orthogonal reflection through the hyperplane $u^{\perp}$. The following diagram illustrates this notion. Note that $z = \frac{B(v,u)}{B(u,u)}u$ below.



Now, let $v, w \in V$. Then, by the above definition of σ_u, we have that

$$\begin{aligned}
B(\sigma_u v, \sigma_u w) &= B\Big(v - 2\frac{B(v,u)}{B(u,u)}u, w - 2\frac{B(w,u)}{B(u,u)}u\Big) \\
&= B(v,w) - 2\frac{B(w,u)}{B(u,u)}B(v,u) - 2\frac{B(v,u)}{B(u,u)}B(u,w) + 4\frac{B(v,u)B(w,u)}{B(u,u)^2}B(u,u) \\
&= B(v,w) - 4\frac{B(v,u)B(w,u)}{B(u,u)} + 4\frac{B(v,u)B(w,u)B(u,u)}{B(u,u)^2} \\
&= B(v,w)
\end{aligned}$$

so $\sigma_u \in O(V)$.

Since $B(u,u) \neq 0$, $W = \langle u \rangle$ is a non-degenerate subspace of V. Then, by Proposition 2.9 in the text, we have that $V = \langle u \rangle \oplus \langle u \rangle^\perp$. Note that $\sigma_u(u) = -u$ and if $u \perp v$ then $\sigma_u(v) = v$. Now set $u_1 = u$ and choose any basis $\{u_2, \ldots, u_n\}$ for $\langle u \rangle^\perp$, then relative to the basis $\{u_1, u_2, \ldots, u_n\}$ for V the matrix representing σ_u is $\begin{bmatrix} -1 & 0 \\ 0 & I_{n-1} \end{bmatrix}$. Since $\det\begin{bmatrix} -1 & 0 \\ 0 & I_{n-1} \end{bmatrix} = -1$, it follows that σ_u is improper, so $\sigma_u \notin SO(V)$.