Orthogonal Geometry

(char $F \neq 2$)

Let $V$ be a quadratic space of dimension $n \geq 2$ over a field $F$, with char $F \neq 2$. Let $B$ be a non-degenerate orthogonal form on $V$. The isometries of $V$ (called orthogonal transformations) comprise the orthogonal group $O(V)$. Thus,

$$O(V) = \{ \tau \in GL(V) \mid B(\tau u, \tau v) = B(u, v), \text{ all } u, v \in V \} \leq GL(V)$$

If $\tau \in V$ and $\{v_1, \ldots, v_n\}$ is a basis for $V$ then $\tau \in O(V)$ if and only if $T^t \hat{B}T = \hat{B}$, where $T$ is the matrix representing $\tau$ relative to the basis.

**Proof.** Case I ($\tau \in O(V) \Rightarrow T^t \hat{B}T = \hat{B}$): Suppose $\tau \in O(V)$. Then, we know that if $u, v \in V$, then $B(\tau u, \tau v) = B(u, v)$. By definition of $\hat{B}$, we know that $B(u, v) = u^t \hat{B}v$. Thus,

$$B(\tau u, \tau v) = (\tau u)^t \hat{B}(\tau v) = u^t T^t \hat{B}Tv. \text{ And, since } B(\tau u, \tau v) = B(u, v), \text{ we know that}$$

$$B(\tau u, \tau v) = u^t T^t \hat{B}Tv = u^t \hat{B}v = B(u, v)$$

We know that $u^t T^t \hat{B}Tv = u^t \hat{B}v$ is true for all vectors $u, v \in V$. We also know that $\hat{B}_{ij}$, the $ij$-th entry of $\hat{B}$, is equal to $e_i^t \hat{B}_{ij} e_j$, where $e_i$ is a basis vector with 1 in the $i$-th place and 0 elsewhere and $e_j$ is a basis vector with 1 in the $j$-th place and 0 elsewhere. Then, $u^t T^t \hat{B}Tv = u^t \hat{B}v$ implies that $\left( (T^t \hat{B}) \right)_{ij} = \hat{B}_{ij}$ for $1 \leq i \leq n$ and $1 \leq j \leq n$. Therefore, since every $ij$-th entry of $T^t \hat{B}$ equals the $ij$-th entry of $\hat{B}$ it follows that $T^t \hat{B}T = \hat{B}$.

Case II ($T^t \hat{B}T = \hat{B} \Rightarrow \tau \in O(V)$): Suppose $T^t \hat{B}T = \hat{B}$. Let $u, v \in V$. Then

$$T^t \hat{B}T = \hat{B} \Rightarrow u^t T^t \hat{B}T = u^t \hat{B}$$

$$\Rightarrow u^t T^t \hat{B}Tv = u^t \hat{B}v$$

$$\Rightarrow (\tau u)^t \hat{B}(\tau v) = u^t \hat{B}v$$

$$\Rightarrow B(\tau u, \tau v) = B(u, v)$$
As a result,

$$T^t \hat{B} T = \hat{B} \Rightarrow \det(T^t \hat{B} T) = \det(\hat{B})$$
$$\Rightarrow \det(T^t) \det(\hat{B}) \det(T) = \det(\hat{B})$$
$$\Rightarrow \det(T)^2 \det(\hat{B}) = \det(\hat{B})$$

Since $B$ is non-degenerate, we know that $\det(\hat{B}) \neq 0$. Therefore, from the preceding expression, we see that $\det(T)^2 = 1$, which implies that $\det T = \pm 1$. Since $\text{char } F \neq 2$, $1 \neq -1$, so we have two cases: (1) $\det \tau = 1$, in which case $\tau$ is called a rotation (or a proper orthogonal transformation); and, (2) if $\det \tau = -1$, in which case $\tau$ is called a reversion (or an improper orthogonal transformation).

The rotations in $O(V)$ form the special orthogonal group $SO(V)$. The special orthogonal group is clearly nonempty because if we let $T = I_n$ be the matrix representing $\tau$ relative to the basis, then we have that $T^t \hat{B} T = \hat{B}$, which implies that $\tau \in O(V)$ (as we showed above). And, since $\det T = \det I_n = 1$, we know that $\tau \in SO(V)$. So, we know that proper orthogonal transformations exist in $O(V)$.

We now show that improper orthogonal transformations exist in $O(V)$. Let $u \in V$, be any vector such that $B(u, u) \neq 0$ (i.e. $u$ is anisotropic). We define a linear transformation $\sigma_u$ to be:

$$\sigma_u(v) = v - 2 \frac{B(v, u)}{B(u, u)} u$$

for all $v \in V$. Defined in this way, $\sigma_u$ is the orthogonal reflection through the hyperplane $u^\perp$. The following diagram illustrates this notion. Note that $z = \frac{B(v, u)}{B(u, u)} u$ below.
Now, let \( v, w \in V \). Then, by the above definition of \( \alpha_u \), we have that

\[
B(\sigma_u v, \sigma_u w) = B(v - 2 \frac{B(v, u)}{B(u, u)} u, w - 2 \frac{B(w, u)}{B(u, u)} u)
\]

\[
= B(v, w) - 2 \frac{B(w, u)}{B(u, u)} B(v, u) - 2 \frac{B(v, u)}{B(u, u)} B(u, w) + 4 \frac{B(v, u) B(w, u)}{B(u, u)^2} B(u, u)
\]

\[
= B(v, w) - 4 \frac{B(v, u) B(w, u)}{B(u, u)} + 4 \frac{B(v, u) B(w, u) B(u, u)}{B(u, u)^2}
\]

\[
= B(v, w)
\]

so \( \sigma_u \in O(V) \).

Since \( B(u, u) \neq 0 \), \( W = \langle u \rangle \) is a non-degenerate subspace of \( V \). Then, by Proposition 2.9 in the text, we have that \( V = \langle u \rangle \oplus \langle u \rangle^\perp \). Note that \( \sigma_u(u) = -u \) and if \( u \perp v \) then \( \sigma_u(v) = v \). Now set \( u_1 = u \) and choose any basis \( \{u_2, \ldots, u_n\} \) for \( \langle u \rangle^\perp \), then relative to the basis \( \{u_1, u_2, \ldots, u_n\} \) for \( V \) the matrix representing \( \sigma_u \) is

\[
\begin{bmatrix}
-1 & 0 \\
0 & I_{n-1}
\end{bmatrix}
\]

Since \( \det \begin{bmatrix}
-1 & 0 \\
0 & I_{n-1}
\end{bmatrix} = -1 \), it follows that \( \sigma_u \) is improper, so \( \sigma_u \notin SO(V) \).