THE MCKAY CORRESPONDENCE

1. McKay correspondence for subgroups of SU(2)

These notes describe some ideas of McKay ([3]). The homework will ask you to show that they describe completely the branching laws from $SU(2)$ to compact subgroups.

The general setting is that we have a compact group G and a finite collection

$$(\pi_1, V_{\pi_1}), \dots, (\pi_r, V_{\pi_r}) \tag{1.1a}$$

of finite-dimensional complex representations of G (not necessarily irreducible). From this we construct a graph Γ. The vertices of Γ are the irreducible representations of G; we think of a vertex ρ as labeled by its dimension $d(\rho)$. The edges of Γ come in different colors, labeled by the representations π_j. The edges are directed, and they are equipped with nonnegative integer multiplicities. The rule is that the edge of color π_j from ρ to τ has multiplicity equal to

$$\text{multiplicity of } \tau \text{ in } \rho \otimes \pi_j = \dim \operatorname{Hom}_G(\tau, \rho \otimes \pi_j). \tag{1.1b}$$

By the complete reducibility of finite-dimensional representations of G, we have

$$\begin{aligned} \dim \operatorname{Hom}_G(\tau, \rho \otimes \pi_j) &= \dim \operatorname{Hom}_G(\rho \otimes \pi_j, \tau) \\ &= \dim \operatorname{Hom}_G(\rho, \tau \otimes \pi_j^*); \end{aligned} \tag{1.1c}$$

the last equality is an associativity formula for tensor and Hom. That is, *the multiplicity of the π_j edge from ρ to τ is equal to the multiplicity of the π_j^* edge from τ to ρ.* From now on we will assume (to simplify the notation) that

$$\text{every } \pi_j \text{ is self-dual.} \tag{1.1d}$$

Then the edges of the McKay graph Γ are undirected. Computing dimensions of tensor products gives the formula

$$\dim(\pi_j) d(\rho) = \sum_{\rho \overset{\pi_j}{\longrightarrow} \tau} d(\tau); \tag{1.1e}$$

what this means is that $d(\tau)$ appears with multiplicity equal to the multiplicity of the edge of color π_j from ρ to τ. More generally, we can compute the trace of any element $x \in G$:

$$\Theta_{\pi_j}(x)\Theta_\rho(x) = \sum_{\rho \xrightarrow{\pi_j} \tau} \Theta_\tau(x). \tag{1.1f}$$

This says that (for fixed x) the vector $\Theta.(x)$ (with entries indexed by the vertices of the McKay graph Γ) is an eigenvector for the π_j-colored adjacency matrix of Γ, with eigenvalue $\Theta_{\pi_j}(x)$. This observation comes from [4].

What McKay understood was that these ideas are particularly simple and beautiful in case G is a compact subgroup of $SU(2)$, and the single representation π is the unique (and therefore self-dual) two-dimensional representation of $SU(2)$, restricted to G.

Theorem 1.2 (McKay [3])**.** *Suppose G is a nontrivial compact subgroup of $SU(2)$, and π is the corresponding two-dimensional representation. Form the McKay graph Γ as in* (1.1)*. That is*

a) *the vertices of Γ are the irreducible representations ρ of G;*
b) *each vertex is labeled by its dimension $d(\rho)$;*
c) *the vertices ρ and τ are joined by m edges if τ appears with multiplicity m in $\rho \otimes \pi$.*

The graph Γ has the following properties.

(1) *(base point) There is a distinguished vertex, labeled* 1*.*
(2) *(harmonic) Each label is twice the sum of the adjacent labels.*
(3) *(connected) The graph is connected.*
(4) *(no loops) No edge connects a vertex to itself.*

Proof. Part (1) refers to the trivial representation of G. Part (2) is (1.1e). Part (3) says that every representation of G appears in some tensor power of π. This is a consequence of the fact that π is faithful and self-dual, and the Stone-Weierstrass approximation theorem.

Part (4) says that ρ cannot occur in $\rho \otimes \pi$. If $-I \in G$, then this is a consequence of the fact that $\pi(-I) = -1$; so suppose $-I \notin G$. By the classification of compact subgroups of $SU(2)$ (that is a *terrible* reason; I would be very happy to hear a better one) G must be abelian (in fact a cyclic group of odd order). Then (4) amounts to the statement that π does not contain the trivial representation of G. But the restriction of π to an abelian subgroup G must be the sum of some one-dimensional character ξ of G and its inverse $-\xi$ (by the determinant one condition on G). Since G is nontrivial, ξ and $-\xi$ must both be nontrivial. $\square$

A (labeled) graph satisfying conditions (1)–(4) of McKay's Theorem 1.2 is called a *harmonic graph.*

Theorem 1.3. *The harmonic graphs of Theorem 1.2 are precisely the graphs of closed subgroups of $SU(2)$: any harmonic graph Γ arises from a compact subgroup G of $SU(2)$ that is unique up to conjugacy.*

I will sketch the proof in class on May 7. The only proof I know is to classify the graphs and to observe that they correspond to the known list of subgroups of $SU(2)$ (due to Felix Klein [2]; see [1], Theorem 5.9.1 (or Theorem 6.12.1 in the second edition)).

2. McKay correspondence and simply laced root data

Recall from the classification of compact Lie groups discussed in class the notion of a *reduced root datum*

$$(X^*, R, X_*, R^\vee). \tag{2.1a}$$

Here X^* and X_* are dual lattices, and $R \subset X^*$ and $R^\vee \subset X_*$ are finite subsets in bijection $\alpha \leftrightarrow \alpha^\vee$. I won't recall all the axioms here. In the notes `repweights.pdf` (and in many other places!) you can find the notion of a set of *positive roots* R^+ and the corresponding *simple roots and coroots*

$$\Pi_0 \subset R^+ \subset R, \qquad \Pi_0^\vee \subset (R^+)^\vee \subset R^\vee. \tag{2.1b}$$

I stated also that the Weyl group $W(R)$ acts in a simply transitive way on choices of positive roots; so all the definitions that follow are (up to the action of W) independent of that choice.

As a fairly easy consequence of the definitions, we have

(1) every root is in R^+ or $-R^+$ but not both; and
(2) a root is in R^+ if and only if it is a nonnegative integer combination of simple roots.

Suppose α and β are *distinct* simple roots. Then the formula for reflections and the axioms for a root system say that

$$s_\alpha(\beta) = \beta - \langle \beta, \alpha^\vee \rangle \alpha \tag{2.1c}$$

is necessarily a root. Since the simple root β appears with coefficient $+1$, we conclude that $s_\alpha(\beta)$ must be a *positive* root, and therefore that

$$-\langle \beta, \alpha^\vee \rangle \in \mathbb{N} \qquad (\alpha \neq \beta \in \Pi_0). \tag{2.1d}$$

Lemma 2.2. *Suppose $\alpha \neq \beta \in \Pi_0$, and that $\langle \beta, \alpha^\vee \rangle \neq 0$. Then one of the two integers $\langle \beta, \alpha^\vee \rangle$ and $\langle \alpha, \beta^\vee \rangle$ is equal to -1, and the other is -1, -2, or -3.*

This lemma is the beginning of the classification theory of root systems, and can be found (more or less) in places like Humphreys' book on Lie algebras and representation theory.

Definition 2.3. *The* Dynkin diagram *of the root datum is the (partially) directed graph Γ_0 with vertex set Π_0. The distinct edges α and β are joined by an edge if and only if $s_\alpha(\beta) \neq \beta$. In that case the edge has multiplicity*

$$\langle \alpha, \beta^\vee \rangle \langle \beta, \alpha^\vee \rangle = 1,\ 2,\ \textit{or}\ 3.$$

If the multiplicity is 2 or 3, then the edge is directed to point toward α when $\langle \alpha, \beta^\vee \rangle = -1$. The root system is called simply laced *if there are no directed (multiple) edges in the Dynkin diagram.*

The connected components of the Dynkin diagram correspond to the simple summands of the root system.

Definition 2.4. *A* lowest root *γ for R^+ is one such that $-\alpha + \gamma$ is not a root for any $\alpha \in R^+$. Lowest roots always exist; there is exactly one for each simple summand of R.*

To simplify the notation, assume now that R is simple, so that the Dynkin diagram Γ_0 is connected, and there is a unique *lowest root γ. Necessarily γ is a negative root:*

$$-\gamma = \sum_{\alpha \in \Pi_0} n_\alpha \alpha \qquad (n_\alpha \in \mathbb{N}).$$

If we define $n_\gamma = 1$, then

$$0 = n_\gamma \gamma + \sum_{\alpha \in \Pi_0} n_\alpha \alpha.$$

The extended Dynkin diagram *has vertex set*

$$\Pi = \Pi_0 \cup \{\gamma\},$$

labeled by the positive integers n_β defined above. Edges are defined exactly as for the ordinary Dynkin diagram. There is one additional case possible: for the root system of type A_1, $\gamma = -\alpha$; and in this case we join α to γ by an undirected double edge.

The definition of the labels n_β now gives

$$\sum_{\beta \in \Pi} n_\beta \beta = 0.$$

If the root system is simply laced, then the extended diagram also has no directed edges (although in type A_1 there is an undirected double edge).

It is easy to define the extended diagram for a non-simple root system: we add one vertex (a lowest root) for each simple factor, and label vertices by considering each simple factor separately.

Proposition 2.5. *Suppose $(X^*, R, X_*, R^\vee)$ is a simply laced simple root datum (meaning that the Dynkin diagram Γ_0 is connected and has no directed edges). Then the extended Dynkin diagram Γ (Definition 2.4 is a harmonic graph in the sense of Theorem 1.2. The distinguished vertex is the lowest root γ, and the labels are given by multiplicities of simple roots in $-\gamma$.*

Proof. That the lowest root is labeled 1 is part of the definition of the labels. The definition of edges gives for any α and β in Π

$$\langle \alpha, \beta^\vee \rangle = \begin{cases} 2, & \alpha = \beta \\ -1, & \alpha \text{ --- } \beta \ \text{adjacent} \\ -2, & \alpha = \beta \ \text{doubly adjacent;} \end{cases} \tag{2.6}$$

the third case is $\beta = -\alpha$, occurring in A_1. The defining property of the labels gives

$$0 = \langle \sum_{\alpha \in \Pi} n_\alpha \alpha, \beta^\vee \rangle.$$

Computing the right side using (2.6) gives

$$0 = 2n_\beta - \sum_{\alpha - \beta} n_\alpha,$$

and this is condition (2) of the McKay theorem. The connectedness condition (3) is the simplicity of R, and the no loops condition (4) is part of the definition of the Dynkin diagram. □

Theorem 2.7 (Garfinkle)**.** *Suppose Γ is a harmonic graph (cf. Theorem 1.2. Then Γ is the extended Dynkin diagram of a simply laced root system; so Γ_0 (the subgraph with the distinguished vertex removed) is the Dynkin diagram.*

Of course the classification of harmonic graphs is fairly easy; so one approach is simply to write down a root datum for each graph. But it's more aesthetically pleasing to argue without the classification. I will try either to talk about such a proof in class, or to add details later. The main ideas are in the exercises in Bourbaki's chapter on Coxeter groups.

There are various ways to deal with non-simply laced root data.

References

[1] Michael Artin, *Algebra*, Prentice Hall Inc., Englewood Cliffs, NJ, 1991.

[2] Felix Klein, *Lectures on the icosahedron and the solution of equations of the fifth degree*, Second and revised edition, Dover Publications Inc., New York, N.Y., 1956. Translated into English by George Gavin Morrice.

[3] John McKay, *Graphs, singularities, and finite groups*, The Santa Cruz Conference on Finite Groups (Univ. California, Santa Cruz, Calif., 1979), Proc. Sympos. Pure Math., vol. 37, Amer. Math. Soc., Providence, R.I., 1980, pp. 183–186.

[4] ———, *Cartan matrices, finite groups of quaternions, and Kleinian singularities*, Proc. Amer. Math. Soc. **81** (1981), no. 1, 153–154.