MANIFOLD STRUCTURES IN ALGEBRA

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1. Definitions

Our aim is to describe the manifold structure on classical linear groups and from there deduce a number of results. Before we begin we make a few fundamental definitions.

Definition 1.1. *Let S be a subset of $\mathbb{R}^m$ and S' be a subset of $\mathbb{R}^n$. A map $f : S \to S'$ is called a homeomorphism if f is continuous and bijective and f^{-1} is continuous.*

Definition 1.2. *A subset S of $\mathbb{R}^m$ is called a manifold of dimension d if every point p of S has a neighborhood in S which is homeomorphic to an open set in $\mathbb{R}^d$.*

Definition 1.3. *A Lie group is a (differentiable) manifold which is also endowed with group structure such that the map $G \times G \to G$ defined by $(x, y) \mapsto xy^{-1}$ is continuously differentiable infinitely many times.*

Definition 1.4. *A Lie algebra V over a field F is a vector space together with a bilinear operator $[,] : V \times V \to V$, called bracket, with the following properties:*

$$[x, y] = - [x, y]$$
$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0.$$

The last of these concepts will propel most of the discussion. It is in working with the Lie algebras of Lie groups that we can more easily determine facts about the groups themselves. Today, we will only be working with the fields $\mathbb{R}$ and $\mathbb{C}$.

2. Manifold Structure on Groups

There are two useful ways to interpret how a Lie algebra relates to a Lie group. Say that G is a Lie group. Then, with $x \in G$, we define *left translation by x* by the map

$$l_x(y) = xy,$$

for all $y \in G$. We digress briefly to discuss what it means for a vector field X to be on a manifold M. Where $C^\infty(M)$ is the set of all infinitely differentiable functions on M, X on M is a mapping $X : C^\infty(M) \to C^\infty(M)$ satisfying the following two axioms:

$$X(af + bg) = aX(f) + bX(g)$$
$$X(fg) = X(f)g + fX(g),$$

where $a, b \in \mathbb{R}$ and $f, g \in C^\infty(M)$. If we now take a vector field X on G we can call X left invariant if for each $x \in G$ the relation

$$dl_x \circ X = X \circ l_x$$

holds. It is valid to consider this relation because left translation by x is defined to be a *diffeomorphism.* That is to say, it is a differentiable map between manifolds whose inverse is also differentiable. The Lie algebra L of G is the set left invariant vector fields. To see this fact we need only verify that the Lie bracket of two left invariant vector fields $X, Y \in L$, defined

$$[X, Y] = XY - YX$$

is also a left invariant vector field and that this definition of the bracket satisfies the axioms for a Lie algebra. The axioms are verified directly and the left invariance of $[X, Y]$ follows plainly from the left invariance of each X and Y.

The following theorem offers an alternative interpretation of Lie algebras.

Theorem 2.1. *Let G be a Lie group and L its set of left invariant vector fields. Then L is a real vector space and the map $\alpha : L \to G_e$ defined by*

$$\alpha(X) = X(e)$$

is an isomorphism of L with the tangent space G_e to G at the identity.

Proof. We omit the proof of L being a vector space and the linearity of α. We have only to prove that α is bijective. Assume $\alpha(X) = \alpha(Y)$. Then for each $x \in G$ we have

$$X(x) = dl_x(X(e)) = dl_x(Y(e)) = Y(x);$$

hence $X = Y$ and α is injective. Now take $z \in G_e$ and let $X(x) = dl_x(z)$ for each $x \in G$. Then $\alpha(X) = z$, and X is left invariant since

$$X(yx) = dl_{yx}(z) = dl_y dl_x(z) = dl_y(X(z))$$

for all $x, y \in G$. Therefore, α is surjective and thus bijective. $\square$

This theorem means that α induces a Lie algebra structure on the tangent space G_e at the identity (i.e. since G_e is isomorphic to L it offers a new interpretation of the Lie algebra of G).

What does all this mean in the context of linear groups? Consider the set $\mathfrak{gl}(n,\mathbb{R})$ of all n by n matrices. If we set $[A,B]=AB-BA$ then $\mathfrak{gl}(n,\mathbb{R})$ becomes a Lie algebra with dimension n^2. We take for granted here that vector spaces are very naturally manifolds, which implies that $\mathfrak{gl}(n,\mathbb{R})$ is a manifold. Hence, the general linear group $GL(n,\mathbb{R})$ inherits manifold structure as an open subset of $\mathfrak{gl}(n,\mathbb{R})$. $GL(n,\mathbb{R})$ is also a Lie group. We verify this fact by first looking at the linear map m_{ij} on $\mathfrak{gl}(n,\mathbb{R})$ which carries each matrix to its ijth entry. Then if $x,y\in GL(n,\mathbb{R})$, $m_{ij}(xy^{-1})$ is a rational function of $\{m_{kl}(x)\}$ and $\{m_{kl}(y)\}$ with non-zero denominator, proving that the map $(x,y)\to xy^{-1}$ is continuously differentiable infinitely many times. $\mathfrak{gl}(n,\mathbb{R})$ is the Lie algebra of $GL(n,\mathbb{R})$. To show this, which we will not do here, we would need to prove there is a Lie algebra isomorphism between $GL(n,\mathbb{R})_e$ and $\mathfrak{gl}(n,\mathbb{R})$. With this very basic foundation we can go on to discuss the information we can glean about classical linear groups.

3. Classical Groups

We present the following theorem without proof in the interest of time.

Theorem 3.1. *Let A be an abstract subgroup of $GL(n,\mathbb{C})$ and let $\mathfrak{a}$ be a subspace of $\mathfrak{gl}(n,\mathbb{C})$. Let U be a neighborhood of 0 in $\mathfrak{gl}(n,\mathbb{C})$ diffeomorphic under the exponential map with a neighborhood V of the identity in $GL(n,\mathbb{C})$. Suppose that*

$$e^{U\cap\mathfrak{a}} = A\cap V.$$

Then A is a Lie subgroup of G, $\mathfrak{a}$ is a subalgebra of $\mathfrak{gl}(n,\mathbb{C})$ and $\mathfrak{a}$ is the Lie algebra of A.

Given this theorem, with some “nice” choices of sets of matrices we can verify that the classical subgroups of $GL(n,\mathbb{C})$ are indeed Lie groups and thus manifolds. Specifically, we need to find sets of matrices, U and V such that the exponential map is a diffeomorphism from U to V. Finding these sets takes some work, but it is possible. After identifying these sets right away with $\mathfrak{a}=\mathfrak{gl}(n,\mathbb{R})\subset\mathfrak{gl}(n,\mathbb{C})$ and $A=GL(n,\mathbb{R})\subset GL(n,\mathbb{C})$ we can see that $GL(n,\mathbb{R})$ is a Lie group, and hence manifold, with Lie algebra $\mathfrak{gl}(n,\mathbb{R})$. The following are further examples of choices of $\mathfrak{a}$ and A that show classical groups are manifolds:

$$\begin{aligned}
\mathfrak{u}(n) &= \{A\in\mathfrak{gl}(n,\mathbb{C}) : A^c + A^t = 0\}\\
\mathfrak{sl}(n,\mathbb{C}) &= \{A\in\mathfrak{gl}(n,\mathbb{C}) : trA = 0\}\\
\mathfrak{o}(n,\mathbb{C}) &= \{A\in\mathfrak{gl}(n,\mathbb{C}) : A + A^t = 0\}.
\end{aligned}$$

and for a corresponding choices of A we have:

$$U(n) = \{A \in GL(n, \mathbb{C}) : A^{-1} = (A^t)^c\}$$
$$SL(n, \mathbb{C}) = \{A \in GL(n, \mathbb{C}) : det A = 1\}$$
$$O(n, \mathbb{C}) = \{A \in GL(n, \mathbb{C}) : A^{-1} = A^t\}$$

where A^c is the complex conjugate of the matrix A.

We will show explicitly the argument that $U(n, \mathbb{C})$ is a closed Lie subgroup of $GL(n, \mathbb{C})$ with Lie algebra $\mathfrak{u}(n, \mathbb{C})$ and note that similar arguments hold for $SL(n, \mathbb{C})$ and $O(n, \mathbb{C})$.

Firstly we need again to identify sets $U \subset \mathfrak{gl}(n, \mathbb{C})$ and $V \subset GL(n, \mathbb{C})$ that are diffeomorphic under the exponential map. U and V must also be closed under transposition, complex conjugation and taking inverses. This is an attainable goal, but because of the limited scope of this presentation we simply note that it is possible to find such sets and assume that we have done so.

If $A \in U \cap \mathfrak{u}(n)$, then $((e^A)^c)^t = e^{(A^c)^t} = e^{-A}$. Accordingly, $((e^A)^c)^t e^A = e^{-A} e^A = I$, which implies that $e^A \in U(n) \cap V$.

Conversely, suppose $A \in U$ and that $e^A \in U(n) \cap V$. Then $e^{-A} = (e^A)^{-1} = ((e^A)^c)^t = e^{(A^c)t}$, which implies that $-A = (A^c)^t$, because $-A$ and $(A^c)^t$ are in U and the exponential map is 1:1 on U. Thus $A \in U \cap \mathfrak{u}(n)$, which finally implies that $U(n)$ is a closed Lie subgroup of $GL(n, \mathbb{C})$ with Lie algebra $\mathfrak{u}(n)$.

It is worth noting that the dimension of a closed Lie subgroup is the same as the dimension of its Lie algebra. As such $U(n)$ has dimension n^2, $SL(n, \mathbb{C})$ has dimension $2n^2 - 2$ and $O(n, \mathbb{C})$ has dimension $n(n-1)$.

The last thing we will do today is use the fact that the orthogonal group $O(n)$ is a closed Lie subgroup of $GL(n, \mathbb{R})$ to impose a manifold structure on the set $M_k(V)$ of all k-dimensional subspaces of V a d-dimensional real vector space (the Grassman variety).

First we choose a basis $\{v_i\}_{i=1}^d$ for V. Then we can see that $O(d)$ acts on V by matrix multiplication. Note that non-singular linear transformations map k-planes to k-planes. Therefore, we have a map $\phi : O(d) \times M_k(V) \to M_k(V)$. Note that for any two k-planes, say R and S, there is an $x \in O(d)$ such that $\phi(x, R) = S$.

Suppose now that R_0 is the k-plane spanned by the first k vectors of the basis chosen in the beginning. Let H be the subset of $O(d)$ that fixes R_0. We see that H is made up of block matrices where the upper left block is an element of $O(k)$, the lower right block is any element of $O(d-k)$ and all other entries are zero.

Therefore, H is a closed subgroup of $O(d)$ which can be identified with $O(k) \times O(d-k)$. Then the map $x(O(k) \times O(d-k)) \mapsto \phi(x, R_0)$ is a

1:1 map of the manifold $\frac{O(d)}{O(k)\times O(d-k)}$ onto the set $M_k(V)$. If we require this map to be a diffeomorphism $M_k(V)$ is made into a manifold. One must simply check that structure is independent.