The orbit method for reductive groups

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Outline

Abstract harmonic analysis

Say Lie group G acts on manifold M. Can ask about

- topology of M
- solutions of G-invariant differential equations
- special functions on M (automorphic forms, etc.)

Method step 1: LINEARIZE. Replace M by Hilbert space $L^2(M)$. Now G acts by unitary operators.

Method step 2: DIAGONALIZE. Decompose $L^2(M)$ into minimal G-invariant subspaces.

Method step 3: REPRESENTATION THEORY. Study minimal pieces: irreducible unitary repns of G.

Difficult questions: how does DIAGONALIZE work, and what do minimal pieces look like?

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Plan of talk

- Outline strategy for decomposing $L^2(M)$: analogy with “double centralizers” in finite-diml algebra.
- Strategy $\rightsquigarrow$ **Philosophy of coadjoint orbits:**

irreducible unitary representations
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(nearly) symplectic manifolds with
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Decomposing a representation

Given: interesting operators $\mathcal{A}$ on Hilbert space $\mathcal{H}$.
Goal: decompose $\mathcal{H}$ in $\mathcal{A}$-invt way.

Finite-dimensional case:

$V/\mathbb{C}$ fin-diml, $\mathcal{A} \subset \mathrm{End}(V)$ cplx semisimple alg of ops.

Classical structure theorem:

$W_1, \ldots, W_r$ list of all simple $\mathcal{A}$-modules; then

$$\mathcal{A} \simeq \mathrm{End}(W_1) \times \cdots \times \mathrm{End}(W_r) \qquad V \simeq m_1 W_1 + \cdots + m_r W_r.$$

Positive integer m_i is *multiplicity* of W_i in V.

Slicker version: define *multiplicity space* $M_i = \mathrm{Hom}_{\mathcal{A}}(W_i, V)$; then $m_i = \dim M_i$, and

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Commuting algebras and all that

Theorem

Suppose $\mathcal{A}$ is semisimple algebras of operators on V as above; define $\mathcal{Z} = \mathrm{Cent}_{\mathrm{End}(V)}(\mathcal{A})$, a second semisimple algebra of operators on V.

1. *Relation between $\mathcal{A}$ and $\mathcal{Z}$ is symmetric:*
$$\mathcal{A} = \mathrm{Cent}_{\mathrm{End}(V)}(\mathcal{Z}).$$
2. *There is a natural bijection between irr modules W_i for $\mathcal{A}$ and irr modules M_i for $\mathcal{Z}$, given by*
$$M_i \simeq \mathrm{Hom}_{\mathcal{A}}(W_i, V), \qquad W_i \simeq \mathrm{Hom}_{\mathcal{Z}}(M_i, V).$$
3. *$V \simeq \sum_i M_i \otimes W_i$ as a module for $\mathcal{A} \times \mathcal{Z}$.*

Example 1: finite G acts left and right on $\mathbb{C}[G]$.
Example 2: S_n and $GL(E)$ act on $V = T^n(E)$.
But those are stories for other days…

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Infinite-dimensional representations

Need framework to study ops on inf-diml V.

Finite-diml $\leftrightarrow$ infinite-diml dictionary

finite-diml V	$\leftrightarrow$	$C^\infty(M)$
repn of G on V	$\leftrightarrow$	action of G on M
$\mathrm{End}(V)$	$\leftrightarrow$	$\mathrm{Diff}(M)$
$\mathcal{A} = \mathrm{im}(\mathbb{C}[G]) \subset \mathrm{End}(V)$	$\leftrightarrow$	$\mathcal{A} = \mathrm{im}(U(\mathfrak{g})) \subset \mathrm{Diff}(M)$
$\mathcal{Z} = \mathrm{Cent}_{\mathrm{End}(V)}(\mathcal{A})$	$\leftrightarrow$	$\mathcal{Z} = G$-invt diff ops

Suggests: G-irreducible pieces of function space correspond to simple modules for G-invt diff ops.

Which differential operators commute with G?

Answer leads to generalizations of dictionary...

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repn of G on V	$\leftrightarrow$	action of G on M
$\mathrm{End}(V)$	$\leftrightarrow$	$\mathrm{Diff}(M)$
$\mathcal{A} = \mathrm{im}(\mathbb{C}[G]) \subset \mathrm{End}(V)$	$\leftrightarrow$	$\mathcal{A} = \mathrm{im}(U(\mathfrak{g})) \subset \mathrm{Diff}(M)$
$\mathcal{Z} = \mathrm{Cent}_{\mathrm{End}(V)}(\mathcal{A})$	$\leftrightarrow$	$\mathcal{Z} =$ G-invt diff ops

Suggests: G-irreducible pieces of function space correspond to simple modules for G-invt diff ops.

Which differential operators commute with G?

Answer leads to generalizations of dictionary. . .

Infinite-dimensional representations

Need framework to study ops on inf-diml V.

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Increasing filtration, $(\mathrm{Diff}_p)(\mathrm{Diff}_q) \subset \mathrm{Diff}_{p+q}$.

Theorem (Symbol calculus)

1. *There is an isomorphism of graded algebras*

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Our story so far...

G acts on $M \leftrightsquigarrow T^*(M)$ Hamiltonian G-space.

G-decomp of $C^\infty(M) \leftrightsquigarrow (\text{Diff } M)^G$-modules.

$(\text{Diff } M)^G \overset{\sigma}{\leftrightsquigarrow} C^\infty(T^*(M))^G \leftrightsquigarrow C^\infty((T^*(M))/G)$.

Hope $C^\infty(M)$ irr $\Leftrightarrow$ G has dense orbit on $T^*(M)$.

Suggests generalization...

Hamiltonian G-cone $X \rightsquigarrow$ graded alg $\text{Poly}(X)$.

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Recall: Hamiltonian G-space X comes with (G-equivariant) moment map $\mu\colon X \to \mathfrak{g}^*$.

Kostant's theorem: homogeneous Hamiltonian G-space = covering of G-orbit on $\mathfrak{g}^*$.

Includes classification of symp homog spaces for G. (Riem homog spaces hopelessly complicated.)

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Case of reductive G is still open.

Actually $(\star)$ is false for connected nonabelian reductive G.

But there are still theorems close to $(\star)$.

Two ways to do repn theory for reductive G:

1. start with coadjt orbit, look for repn. Hard.
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Reductive Lie group G $=$ closed subgp of $\underbrace{GL(n,\mathbb{R})}_{\text{main ex}}$

s.t. G closed under transpose, and $\#G/G_0 < \infty$.

From now on G is reductive.

$\mathrm{Lie}(G) = \mathfrak{g} \subset n \times n$ matrices. Bilinear form

$T(X,Y) = \mathrm{tr}(XY) \Rightarrow \mathfrak{g} \overset{G\text{-eqvt}}{\simeq} \mathfrak{g}^*$

Orbits of G on $\mathfrak{g}^* \subset$ conjugacy classes of matrices.

Orbits of $GL(n,\mathbb{R})$ on $\mathfrak{g}^* =$ conj classes of matrices.

Family of orbits: for real numbers λ_1 and λ_{n-1},

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Base point in family:

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Reductive Lie group G $=$ closed subgp of $\underbrace{GL(n,\mathbb{R})}_{\text{main ex}}$

s.t. G closed under transpose, and $\#G/G_0 < \infty$.

From now on G is reductive.

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V n-diml real. $G = GL(V)$ acts on $M = \mathbb{P}V$ = lines in V.

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This hope *does* disappoint us: $C^\infty(M) \supset$ constants, so rep is reducible. Also there's no G-invt msre on M, so no unitary Hilbert space version $L^2(M)$.

Fix both problems: $\delta^{1/2}$ = half-density bdle on $\mathbb{P}V$.

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Family of irr unitary representations

Natural generalization: replace functions on $M = \mathbb{P}V$ by sections of Hermitian line bundle.

Two natural ($GL(V)$-eqvt) real bdles on $\mathbb{P}V$: tautological line bdle $\mathcal{L}$ (fiber at line L is L); and $\mathcal{Q}$ ($(n-1)$ -diml real bundle, fiber at L is V/L).

Given real parameters λ_1 and λ_{n-1}, get Hermitian line bundle $\mathcal{H}(\lambda_1, \lambda_{n-1}) = \mathcal{L}^{i\lambda_1} \otimes (\wedge^{n-1}\mathcal{Q})^{i\lambda_{n-1}}$.

Define

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These are irr unitary representations of $GL(V)$; naturally assoc to coadjt orbits $\mathcal{O}(\lambda_1, \lambda_{n-1})$.

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And now for something completely different. . .

V $2m$-dimensional real vector space, $G = GL(V)$. Fix real $t_m \geq 0$, real s_m, define coadjt orbit

$$\mathcal{O}(s_m + it_m, s_m - it_m) = \{A \in \mathrm{End}(V) \mid \text{ eigval } s_m \pm it_m \text{ mult } m\}.$$

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Parameter s_m corresponds to twisting by one-diml char of $GL(V)$: cumbersome and dull. So pretend it doesn't exist.

Corresponding repns related to cplx alg variety

$$X = \text{ complex structures on } V, \qquad \dim X = m^2.$$

Have K-invariant projective subvariety

$$Z = \text{ orthogonal cplx structures} \quad \dim Z = (m^2 - m)/2 = s.$$

Turns out (Schmid, Wolf) X is $(s+1)$-complete, which means Stein away from Z.

X has G-invt indef Kähler structure, signature $((m^2 - m)/2, (m^2 + m)/2)$; underlying real symplectic mfld is $\mathcal{O}(it_m, -it_m)$ (any $t_m > 0$).

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Base point in family:

$$\mathcal{O}_{m,m} = \{A \in \mathrm{End}(V) \text{ nilpotent, Jordan blocks } m, m\}.$$

Corresponding repns related to cplx alg variety

$$X = \text{ complex structures on } V, \qquad \dim X = m^2.$$

Have K-invariant projective subvariety

$$Z = \text{ orthogonal cplx structures} \quad \dim Z = (m^2 - m)/2 = s.$$

Turns out (Schmid, Wolf) X is $(s+1)$-complete, which means Stein away from Z.

X has G-invt indef Kähler structure, signature $((m^2 - m)/2, (m^2 + m)/2)$; underlying real symplectic mfld is $\mathcal{O}(it_m, -it_m)$ (any $t_m > 0$).

And now for something completely different...

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Actually can also twist by character $|\det|^{is_m}$ of $GL(V)$, but we're pretending that doesn't exist.

Canonical bdle is $\omega_X = \mathcal{L}^{-2m}$.

Very rough idea: $\mathcal{O}(it_m, -it_m) \rightsquigarrow$ repn $\Gamma(\mathcal{L}^p)$.

Better: $\mathcal{O}(it_m, -it_m) \rightsquigarrow$ repn $H^{0,s}(X, \mathcal{L}^p)$.

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Inclusion of compact subvariety Z gives lowest $O(V)$-type: $(t_m + 1)$-Cartan power of $\bigwedge^m(V)$. (Shift $+1$ since $\omega_Z = \omega_X^{1/2} \otimes \mathcal{L}^{-1}$.)

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Problem: holomorphic sections (case $m > 1$) exist only for $p \geq 0$; they extend to cplx Grassmannian of m-planes in $V_{\mathbb{C}}$, give fin diml (non-unitary) rep of $GL(V)$.

Better: $\mathcal{O}(it_m, -it_m) \leftrightsquigarrow$ repn $H^{0,s}(X, \mathcal{L}^p)$.

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Good news: infinitesimally unitary for suff neg $\mathcal{L}$.

Bad news: cond on p complicated, Hilb space is tiny (dense) subspace.

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Parallel techniques deal with elliptic coadjt orbits (that is, orbits of semisimple matrices with purely imaginary eigenvalues.

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What we learn: interesting orbits are those with $t_m + m \in \mathbb{Z}$.

These are *admissible orbits.* (Duflo).

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This is Serre duality plus analytic results of Hon-Wai Wong.

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Quantizing nilpotent orbits

For $GL(n, \mathbb{R})$, nilp coadjt orbits $=$ special points in families of semisimp orbits $\rightsquigarrow$ quantize by continuity (see below).

For other reductive groups, not true: many nilpotent orbits have no deformation to semsimple orbits.

Kostant-Rallis idea: nilp coadjt orbit $\mathcal{O}_{\mathbb{R}}$ has natural K-invt cplx structure $\mathcal{O}_\theta \rightsquigarrow$ holomorphic action of $K_{\mathbb{C}}$.

Get repn of K that quantizes $\mathcal{O}_\theta$; look for a way to extend it to G. Carried out by Brylinski and Kostant for minimal coadjt orbit in many cases.

Rossi-Vergne idea: Given semisimple orbits, quantizations

$$\{\mathcal{O}(\lambda) \mid \lambda \text{ dom reg}\} \rightsquigarrow \{\pi(\lambda) \mid \lambda \text{ dom reg adm}\}.$$

Repns make sense (but may not be unitary) for "all" admissible λ (not dominant or regular).

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Meaning of it all

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After that the class picked up speed.

Many years later I took another class from Bert. At some point he needed differential operators. So he gave an introduction: "You form the algebra generated by the derivations of C^∞."

That's Bert: mathematics at Mach 2, always exciting, and the explanations are always complete; you'll figure them out eventually. The first third of a century has been fantastic, and I hope to keep listening for a very long time.

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