Realizing smooth representations

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Outline

Gelfand's abstract harmonic analysis

Lie group G acts on manifold X, have questions about X.

Step 1. Attach to X Hilbert space $\mathcal{H}(X)$ (e.g. $L^2(X)$). Questions about $X \rightsquigarrow$ questions about $\mathcal{H}(X)$.

Step 2. Find finest G-invt decomp $\mathcal{H} = \oplus_\alpha \mathcal{H}_\alpha$. Questions about $\mathcal{H}(X) \rightsquigarrow$ questions about each $\mathcal{H}_\alpha$.

Each $\mathcal{H}_\alpha$ is irreducible unitary representation of G: indecomposable action of G on a Hilbert space.

Step 3. Understand $\widehat{G}_u =$ all irreducible unitary representations of G: unitary dual problem.

Step 4. Answers about irr reps $\rightsquigarrow$ answers about X.

Today: technical problems in Steps 1 and 3...

Say Question $\leftrightsquigarrow$ eigenfns of G-invt diff op Δ_X.

Problem with Step 1: eigenfunctions not in $L^2(X)$.

Problem with Step 3: try $\mathbb{H}_\alpha =_{\text{def}}$ eigenspace of Δ_X. But no Hilbert space structure.

How to address these problems

Problems arise because eigenspace

$$V_\lambda = \{f \in C^{-\infty}(X) \mid \Delta_X f = \lambda f\}$$

is inconveniently large.

For example, can't complete V_λ to Hilbert space without imposing addl growth conditions on f.

Solution: consider instead dual space

$$V^\lambda = C_c^\infty(X)/(\Delta_X - \lambda)C_c^\infty(X).$$

Wasn't that easy?

One concrete example

$G = U(p,q)$ group of Herm form $\langle , \rangle_{p,q}$ on $\mathbb{C}^n$.

$$Y^n = Y = \{\text{complete flags in } \mathbb{C}^n\} \qquad (\dim_{\mathbb{C}}(Y^n) = \tbinom{n}{2})$$
$$= \{0 = E_0 \subset E_1 \subset \cdots \subset E_n = \mathbb{C}^n \mid \dim E_i = i\}$$

Open orbits of $U(p,q)$ on $Y \leftrightarrow$

$\{(p_i, q_i) \mid p_i + q_i = i,\ p_i \text{ incr}, q_i \text{ incr}\} \leftrightarrow$

$\{S \subset \{1,2,\ldots,n\} \mid |S| = p\}$

$X_S =$ open orbit corr to $S \subset \{1,\ldots,n\}$.

Complex mfld X_S has cplx $K = U(p) \times U(q)$ orbit

$Z_S = \{(E_i) \in X_S \mid E_i = (E_i \cap \mathbb{C}^p) \oplus (E_i \cap \mathbb{C}^q)\} \simeq Y^p \times Y^q$.

Orbit method: $\mathcal{L} \to X_S \overset{?}{\rightsquigarrow}$ unitary reps of G.

Problem. Cpt cplx $Z_S \subset X_S \implies X_S$ not Stein; not enough holom secs of $\mathcal{L}$.

Soln: Dolbeault cohom $H^{0,s}(X_S, \mathcal{L})$, $s = \dim_{\mathbb{C}}(Z_S)$.

Problem. Dolbeault cohom too big for invt Herm form.

Soln: see below, we hope.

Classical Fourier analysis

$G = \mathbb{R}$ acts on $\mathbb{R}$ by translation, $\mathcal{H} = L^2(\mathbb{R})$, $\Delta_X = \frac{d}{dt}$.

Eigenspace representation

$$V_\lambda = \{T \in C^{-\infty}(\mathbb{R}) \mid \frac{dT}{dt} = \lambda T\}$$

Of course V_λ is one-diml, basis $e^{\lambda t}$; never in $L^2(\mathbb{R})$.

Consider instead dual space

$$V^\lambda = C_c^\infty(\mathbb{R}) / \left\{ \frac{d\phi}{dt} - \lambda\phi \mid \phi \in C_c^\infty(\mathbb{R}) \right\}.$$

Subspace by which we divide is equal to

$$\left\{ \psi \in C_c^\infty(\mathbb{R}) \mid \int_{\mathbb{R}} \psi(t) e^{t\lambda}\, dt = 0 \right\},$$

so closed; so topology on V^λ Hausdorff.

First advantage: have trivially quotient map

$$\widehat{\ }(\lambda)\colon C_c^\infty(\mathbb{R}) \to V^\lambda, \quad \phi \mapsto \widehat{\phi}(\lambda)\colon$$

this is Fourier trans at λ on dense $C_c^\infty \subset L^2$.

Unitary structure

$$V^\lambda = C_c^\infty(\mathbb{R})/\left\{\frac{d\phi}{dt} - \lambda\phi \mid \phi \in C_c^\infty(\mathbb{R})\right\}$$

(pre)Unitary structure is $\mathbb{R}$-invt Hermitian form

$$\langle,\rangle^\lambda \colon V^\lambda \times V^\lambda \to \mathbb{C}.$$

Schwartz kernel theorem: such pairing (lifted to $C_c^\infty \times C_c^\infty$) given by distn kernel

$$K_\lambda \in C^{-\infty}(\mathbb{R}\times\mathbb{R}), \quad \langle\phi,\psi\rangle^\lambda = \int_{\mathbb{R}\times\mathbb{R}} K_\lambda(s,t)\phi(s)\overline{\psi(t)}.$$

Form descends to $V^\lambda \iff \frac{\partial K_\lambda}{\partial s} = \lambda K, \quad \frac{\partial K_\lambda}{\partial t} = \overline{\lambda} K.$

Soln is $K_\lambda(s,t) = Ce^{\lambda s}e^{\overline{\lambda}t}\,ds\,dt$, but we don't care.

Form transl invt $\iff K_\lambda(s+x,t+x) = K_\lambda(s,t)$.

Transl invt compatible with diff eqn $\iff \lambda + \overline{\lambda} = 0$; then it's consequence of diff eqn.

Plancherel theorem

$$V^\lambda = C_c^\infty(\mathbb{R}) / \left\{ \frac{d\phi}{dt} - \lambda\phi \mid \phi \in C_c^\infty(\mathbb{R}) \right\} \quad (\lambda \in \mathbb{C})$$

Fourier transform $\widehat{\ }(\lambda) \colon C_c^\infty(\mathbb{R}) \to V^\lambda$.

If $\lambda \in i\mathbb{R}$ there's invt Herm form $\langle , \rangle^\lambda$ on V^λ; normalize kernel to be $ds\, dt$ near identity.

Plancherel theorem:

$$\int_{\mathbb{R}} |\phi(t)|^2 \, dt = c \int_{i\mathbb{R}} \langle \widehat{\phi}(\lambda), \widehat{\phi}(\lambda) \rangle d\lambda.$$

Laplacian on $\mathbb{H}^2$

$$X = \{z \in \mathbb{C} \mid |z|^2 < 1\}, \qquad ds^2 = (1 - |z|^2)^{-2}(dx^2 + dy^2)$$

unit disk model of two-diml hyperbolic space.

$$G = SU(1,1) = \left\{ \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} \mid |\alpha|^2 - |\beta|^2 = 1 \right\} \subset SL(2, \mathbb{C})$$

acts on X by linear fractional transformations:

$$g \cdot z = \frac{\alpha z + \beta}{\overline{\beta} z + \overline{\alpha}} \qquad \left(z \in X,\ g = \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} \in G \right).$$

Laplace-Beltrami operator commutes with action of G:

$$\Delta_X = (1 - |z|^2)^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \quad (z = x + iy \in X)$$

Long tradition of studying eigenspaces

$$V_\lambda = \{T \in C^{-\infty}(X) \mid (\Delta_X - \lambda)T = 0\} \quad (\lambda \in \mathbb{C}).$$

E.g. V_0 = harm fns on X. Always V_λ = repn of G.

Traditional boundary values

Alg homog space $X = G/H \rightsquigarrow$ proj alg $\overline{X} = X \cup \partial X$.

(noncpt analysis on X) $\rightsquigarrow$ (easier cpt analyis on $\overline{X}$).

Our $X \simeq \left\{ \text{lines } \begin{pmatrix} z \\ 1 \end{pmatrix} \mid |z|^2 - 1 < 0 \right\}$ compactifies to

$\overline{X} \simeq \left\{ \text{lines } \begin{pmatrix} z \\ 1 \end{pmatrix} \mid |z|^2 - 1 \leq 0 \right\}$, $\partial X \simeq \left\{ \begin{pmatrix} e^{i\theta} \\ 1 \end{pmatrix} \right\}$.

Idea of bdry value map: eigenfn $\phi(z) \in V_\lambda$ on $X \rightsquigarrow$

$$\phi^\infty(e^{i\theta}) = \lim_{r\to 1} c_\lambda(r)\phi(re^{i\theta}).$$

Problem: limit is terrible (hyperfunction); certainly doesn't exist pointwise, except under strong addl hyps on eigenfn ϕ.

Eigenspace too big.

Traditional harmonic analysis...

Describe eigenspaces like $V_0 =$ harmonic fns on disk

Poisson kernel $P_0(z, e^{i\theta}) = \frac{1-|z|^2}{|z-e^{i\theta}|^2}$ writes harmonic $\phi(z)$ in terms of bdry value $\phi^\infty(\theta)$:

$$\phi(z) = \int_0^{2\pi} P_0(z, e^{i\theta})\phi^\infty(\theta)\, d\theta.$$

Reason: $P_0(\cdot, e^{i\theta})$ is harmonic with bdry value $\delta_{e^{i\theta}}$.

Fourier analysis (using radially symm eigenfn ϕ_λ)

$$\widehat{f}(\lambda, hK) = \int_X f(gK)\phi_\lambda(g^{-1}h)\, d(gK) \in V_\lambda$$

extracts from nice f on X its "λ-eigenvalue part."

Fourier synthesis reassembles (using Plancherel measure $d\mu(\lambda)$):

$$f(gK) = \int_{\text{some } \lambda} \widehat{f}(\lambda, gK) d\mu(\lambda).$$

Casting out eigenspaces

Replace eigenspace V_λ by (smaller!) dual space
$$V^\lambda = C_c^\infty(X, dx)/(\Delta_X - \lambda)C_c^\infty(X, dx).$$
1st gain: Fourier analysis is trivial:
$$\widehat{\phi}(\lambda, \cdot) = \phi + (\Delta_X - \lambda)C_c^\infty(X, dx) \quad (\phi \in C_c^\infty(X, dx)).$$
2nd gain: V^λ has G-invt sesq form. Reason...

Any sesq pairing lifts to $C_c^\infty(X, dx) \times C_c^\infty(X, dx)$; comes from Schwartz distribution kernel:
$$\langle \phi, \psi \rangle^\lambda = \int_{(s,t)\in X\times X} K^\lambda(s, t)\phi(s)\overline{\psi}(t).$$

Condition for pairing to descend to V^λ:
$$(\Delta_X(s) - \lambda)K^\lambda(s, t) = 0, \quad (\Delta_X(t) - \overline{\lambda})K^\lambda(s, t) = 0.$$

Cond for G-invariance of pairing: $K^\lambda(g\cdot s, g\cdot t) = K^\lambda(s, t)$.

THM. V^λ has invt sesq form iff $\lambda = \overline{\lambda}$; normalize $K^\lambda(gK, hK) = \phi_\lambda(g^{-1}h)$ by $K^\lambda(0, 0) = 1$.

Again ϕ_λ is unique radial eigenfn with $\phi_\lambda(0) = 1$.

Holomorphic line bundles on X

Isotropy at 0 for $g \cdot z = \frac{\alpha z+\beta}{\overline{\beta}z+\overline{\alpha}}$ action on unit disk is

$$K = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \overline{\alpha} \end{pmatrix} \mid |\alpha|^2 = 1 \right\}$$

$$\subset G = \left\{ \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} \mid |\alpha|^2 - |\beta|^2 = 1 \right\}.$$

So $X = \{z \in \mathbb{C} \mid |z|^2 < 1\} \simeq G/K$.

For $n \in \mathbb{Z}$, eqvt line bundle

$$\mathcal{L}_n \to X, \quad C^\infty(X, \mathcal{L}_n) \simeq \{f \in C^\infty(G) \mid f(gk) = \alpha^{-n} f(g)\}.$$

Right action of complexified Lie algebra element $E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ defines Cauchy-Riemann operator

$$\overline{\partial} \colon C^\infty(X, \mathcal{L}_n) \to C^\infty(X, \mathcal{L}_{n+2}).$$

Get rep of G on holomorphic sections of $\mathcal{L}_n$

$$W_n = \{T \in C^{-\infty}(X, \mathcal{L}_n) \mid \overline{\partial} T = 0\}.$$

$W_n \rightsquigarrow$ discrete spectrum of Laplacian on $\mathcal{L}_{n+2k}$.

For that, need (pre)Hilbert space structure on W_n...

Traditional holomorphic discrete series

$$X \simeq \left\{ \text{lines } \mathbb{C}\begin{pmatrix} z \\ 1 \end{pmatrix} \right\} \subset \mathbb{CP}^1, \; \mathcal{L}_n(z) = \left[\mathbb{C}\begin{pmatrix} z \\ 1 \end{pmatrix}\right]^{\otimes^n}.$$

$\mathcal{L}_n$ has nowhere zero holom sec $\tau_n(z) = \begin{pmatrix} z \\ 1 \end{pmatrix}^{\otimes^n}$.

So holom secs of $\mathcal{L}_n$ = (holom fns)$\cdot \tau_n$: $W_n \simeq W_0 \cdot \tau_n$.

$$\begin{aligned}
(g \cdot \tau_n)(z) &=_{\text{def}} g \cdot (\tau_n(g^{-1} \cdot z)) \\
&= g \cdot \begin{pmatrix} \frac{\overline{\gamma} z - \delta}{-\overline{\delta} z + \gamma} \\ 1 \end{pmatrix}^{\otimes^n} \quad \left(g = \begin{pmatrix} \gamma & \delta \\ \overline{\delta} & \overline{\gamma} \end{pmatrix}\right) \\
&= (-\overline{\delta} z + \gamma)^{-n} \cdot g \cdot \begin{pmatrix} \overline{\gamma} z - \delta \\ -\overline{\delta} z + \gamma \end{pmatrix}^{\otimes^n} \\
&= (-\overline{\delta} z + \gamma)^{-n} \cdot \begin{pmatrix} z \\ 1 \end{pmatrix}^{\otimes^n} = (-\overline{\delta} z + \gamma)^{-n} \cdot \tau_n(z).
\end{aligned}$$

$W_n \simeq W_0 \cdot \tau_n \rightsquigarrow$ (rep on W_n) $\simeq$ (*multiplier* rep on W_0):

$$(g \cdot_n f)(z) = (-\overline{\delta} z + \gamma)^{-n} f(g^{-1} \cdot z) \quad (f \text{ holom on } X).$$

Unitary structure on W_n

Seeking unitary representation of G related to W_n.

G-invt Herm str on $\mathcal{L}_n$, $\left\| \begin{pmatrix} z \\ 1 \end{pmatrix}^{\otimes n} \right\|^2 = (1-|z|^2)^n$.

Unitary structure on W_n (?):

$$\|\tau\|^2 \overset{?}{=} \int_X \|\tau(z)\|^2 \frac{dz\, d\overline{z}}{(1-|z|^2)^2} \qquad (\tau \in W_n).$$

Using $W_n \simeq W_0 \cdot \tau_n$, rewrite as

$$\|f\tau_n\|^2 \overset{?}{=} \int_X |f(z)|^2 (1-|z|^2)^{n-2}\, dz\, d\overline{z} \quad (f \text{ holomorphic}).$$

Polar coords $\rightsquigarrow \int_0^1 (1-r)^{n-2}\, dr$.

Problem: if $n \leq 1$, converges only for $f = 0$.

Problem: *never* converges for all f.

Solution: (Hermitian) DUAL SPACE...

$$W^n = C_c^\infty(X, \mathcal{L}_n)/\overline{\partial}(C_c^\infty(X, \mathcal{L}_{n+2})).$$

Denominator is densities vanishing on hol secs of $\mathcal{L}_n$; $\exists$ lots because of Cauchy integral formula, etc.

Unitary structure on W^n

$$W^n = C_c^\infty(X, \mathcal{L}_n)/\partial(C_c^\infty(X, \mathcal{L}_{n+2})).$$

Sesq pairing on $W^n \leftrightsquigarrow$ Schwartz distn kernel
$K^n(s,t) \in C^{-\infty}(X \times X, \mathcal{L}_{-n} \times \mathcal{L}_n)$

$$\langle \phi, \psi \rangle^n = \int_{X \times X} K^n(s,t)\phi(s)\overline{\psi(t)}\, ds\, dt.$$

Pairing $\downarrow W^n \iff K^n$ antiholom in s, holom in t.

Pairing G-invt $\iff K^n(g \cdot s, g \cdot t) = K^n(s,t)$.

$\rightsquigarrow$ fn κ on $G \leftrightsquigarrow$ hol sec of $\mathcal{L}_n$ on right and left.

Unique soln corrs to nowhere zero sec τ_n:

$$\kappa^n \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} = \alpha^{-n}$$

THM. W^n has invt sesq form; $K^n(gK, hK) = \tau_n(g^{-1}k)$.
Form is pos def if $n > 0$; semidef if $n \geq 0$.

Unitary rep $W^n \rightsquigarrow$ disc spec of $\mathcal{L}_n \to X \iff n > 1$.

Where can you go from here?

G reductive Lie $\supset L = G^T$, cpt torus T.

$X = G/L$ cplx mfld; Cauchy-Riemann eqns $\leftrightsquigarrow$

$$\mathfrak{u} = \text{pos eigspaces of } T \subset \mathfrak{g}_{\mathbb{C}}.$$

Holom bdle $\mathcal{E} \to X$ $\leftrightsquigarrow$ smooth rep E of L.

Traditional rep of G: Dolbeault cohom of $\mathcal{E}$:

$$W_E = H^{0,p}(X, \mathcal{E}) = H^p(\mathfrak{u}, C^{-\infty}(G, E)^L)$$

Often unitary E for $L \rightsquigarrow$ "ought-to-be-unitary" W_E.
Problem: W_E too big to carry invt sesq form.

Solution: (Hermitian) DUAL SPACE...

$$W^{E^h} = H^{n,n-p}_{\text{cpt}}(X, \mathcal{E}^h) = H_p(\overline{\mathfrak{u}}, C^\infty_c(G, E^h)_L)$$

Easy: unitary $E \rightsquigarrow$ invt Herm form on W^E. OPEN:

Geometric proof of UNITARITY of W^E.

FOURIER TRANSFORM: $C^\infty_c(G/H) \xrightarrow{?} W^E$.

HARMONIC ANALYSIS: e.g., $W^E \overset{?}{\hookrightarrow} L^2(G)$)

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