On the definition of induced representations

1. Introduction. Darij Grinberg asked in class Wednesday about why the definition of induced representation took the shape that it did. This is a huge and awkward question about which one could write huge and awkward books ([KV]), but here are a few quick comments.

Start with a group G and a subgroup H; eventually they'll be topological or Lie groups, but finite is a good place to start. There is an abelian category

$$\mathrm{Rep}(G) = \text{vector spaces } V_\pi \text{ endowed with representation } \pi \text{ of } G,$$

and similarly for H. The most obvious functor is the forgetful functor

$$\mathrm{For}\colon \mathrm{Rep}(G) \to \mathrm{Rep}(H)$$

which leaves the vector space in place and forgets everything about π except its restriction to H. This is an exact functor.

Category theory (or the examples that lie under it) says that the notion of *adjoint functor* is important. Two functors

$$S\colon \mathcal{A} \to \mathcal{B}, \qquad T\colon \mathcal{B} \to \mathcal{A}$$

are said to be *adjoint* if we are given a natural isomorphism

$$\mathrm{Hom}_{\mathcal{B}}(SA, B) \simeq \mathrm{Hom}_{\mathcal{A}}(A, TB).$$

(Precisely, S is called a *left adjoint of* T, and T a *right adjoint of* S. I won't say anything about general existence and uniqueness theorems for adjoints, although this is a very interesting topic.

The big idea is

induction from H to G should be an adjoint of the forgetful functor.

This idea gets the name "Frobenius reciprocity" in representation theory. Notice that I didn't say on which side the adjointness holds; the answer is that it depends on what you want.

2. Example of modules over rings. The (possibly more familiar) situation in module theory starts with a ring homomorphism

$$R_{\mathrm{little}} \to R_{\mathrm{big}},$$

and the corresponding forgetful functor

$$\mathrm{For}\colon R_{\mathrm{big}}\text{-mod} \to R_{\mathrm{little}}\text{-mod}.$$

carrying modules for the big ring to modules (with the same underlying abelian group) for the little ring.

Perhaps the most familiar adjoint to For is the extension of scalars functor

$$\mathbf{Ten}\colon R_{\text{little}}\text{-mod} \to R_{\text{big}}\text{-mod}, \quad \mathbf{Ten}(N) = R_{\text{big}} \otimes_{R_{\text{little}}} N.$$

This functor is a *left* adjoint to For:

$$\mathrm{Hom}_{R_{\text{big}}}(\mathbf{Ten}(N), M) \simeq \mathrm{Hom}_{R_{\text{little}}}(N, \mathrm{For}(M)).$$

The isomorphism identifies ψ on the right with Ψ on the left by the formulas

$$\Psi(r \otimes n) = r \cdot \psi(n), \qquad \psi(n) = \Psi(1_{R_{\text{big}}} \otimes n).$$

You can think of $\mathbf{Ten}(N)$ as a "smallest" extension of N to an R_{big} module. As a left adjoint of the exact functor For, **Ten** is automatically right exact.

But there is another extension of scalars functor

$$\mathbf{Hom}\colon R_{\text{little}}\text{-mod} \to R_{\text{big}}\text{-mod}, \quad \mathbf{Hom}(N) = \mathrm{Hom}_{R_{\text{little}}}(R_{\text{big}}, N).$$

Here the Hom is regarded as a module for R_{big} by the right action of R_{big} on itself. This functor is a *right* adjoint to For:

$$\mathrm{Hom}_{R_{\text{big}}}(M, \mathbf{Hom}(N)) \simeq \mathrm{Hom}_{R_{\text{little}}}(\mathrm{For}(M), N).$$

The isomorphism identifies a map ϕ on the right with Φ on the left by the formulas

$$\Phi(m)(r) = \phi(r \cdot m), \qquad \phi(m) = \Phi(m)(1_{R_{\text{big}}}).$$

You can think of $\mathbf{Hom}(N)$ as a "fat" extension of N to an R_{big}-module. As a right adjoint of the exact functor For, **Hom** is automatically left exact.

Example: two-torsion. Take $R_{\text{little}} = \mathbb{Z}$, so that R_{little}-mod is the category of abelian groups. Take R_{big} to be $\mathbb{Z}/2\mathbb{Z}$, so that R_{little}-mod is the category of abelian groups in which every element has order two. The two extension functors (from abelian groups to two-torsion abelian groups) are

$$\mathbf{Ten}(N) = N/2N, \qquad \mathbf{Hom}(N) = \text{elements of order 2 in } N.$$

In this example you can see various things going on: that the two extension functors are quite different; that either one can be zero even if the other is not; and that there is no natural map from the "small" one $\mathbf{Ten}(N)$ to the "large" one $\mathbf{Hom}(N)$. (In this case there *is* a natural map in the other direction, but that map has no analog for more general ring homomorphisms.)

3. Group representations again. If G is finite, write $\mathbb{C}G$ for the group algebra, so that we can identify

$$\mathrm{Rep}(G) \simeq \mathbb{C}G\text{-mod} \qquad (G \text{ finite}).$$

The first extension functor is

$$\mathbf{Ten}(N) = \mathbb{C}G \otimes_{\mathbb{C}H} N \simeq \{N\text{-valued measures on } G\}/(\text{right translation by } H).$$

The quotient operation in the definition of the tensor product identifies a measure with each of its right translates by H.

The second extension functor is

$$\mathbf{Hom}(N) = \mathrm{Hom}_H(\mathbb{C}G, N) \simeq \{N\text{-valued functions on } G\}^{H\text{-invt on right}}.$$

In the case of finite G, these two functors are naturally equivalent. Why that is true is a good problem to think about.

For general topological G and H, the easy one to generalize is the second:

$$\mathrm{Ind}_H^G(N) = \{f\colon G \to N \text{ continuous} \mid f(gh) = h^{-1} \cdot f(g)\}.$$

This is the definition I used for parabolic induction in class (except that I used smooth functions instead of continuous functions). Frobenius reciprocity (the adjointness property) is

$$\mathrm{Hom}_G(M, \mathrm{Ind}_H^G(N)) \simeq \mathrm{Hom}_H(\mathrm{For}(M), N).$$

This isomorphism identifies Φ on the left with ϕ on the right, by the formulas

$$\Phi(m)(g) = \phi(g^{-1} \cdot m), \qquad \phi(m) = \Phi(m)(1).$$

This is induction functor is actually exact (not only left exact, as the adjointness guarantees).

There is a very useful variant of the induced representation:

$$\mathrm{cInd}_H^G(N) = \{f\colon G \to N \text{ cont, cpt supp mod } H \mid f(gh) = h^{-1} \cdot f(g)\}.$$

That is, we have made the representation space smaller by considering only functions of compact support in G/H. (Since the support of a function in $\mathrm{Ind}_H^G(N)$ is obviously invariant under right translation by H, it makes sense to consider the support as a subset of the homogeneous space.) This version does not satisfy any obvious analogue of the Frobenius reciprocity relation above.

I am not certain how to define a right adjoint to For. For locally compact G, my guess is that something like

$$\mathrm{RInd}_H^G(N) = \{N\text{-valued Borel msrs of cpt supp on } G\}/\overline{(\text{rt trans by } H)}$$

will work. This is a space of "coinvariants of H," the largest quotient on which H acts trivially on the right. Life is complicated by the fact that what you divide by need not (I think) be closed. The Frobenius reciprocity or adjointness relation

$$\mathrm{Hom}_G(\mathrm{RInd}_H^G(N), M) \simeq \mathrm{Hom}_H(N, \mathrm{For}(M))$$

is what I have not completely understood. We want to identify some Ψ on the left with ψ on the right. I think that (given Ψ) you can define

$$\psi(n) = \Psi(n \otimes \delta_e),$$

the image of the measure assigning to $S \subset G$ the measure n if the identity is in S and 0 otherwise. To go the other way is the hard part; so suppose we have ψ on the right, and we want to define Ψ on the right. So suppose the δ is an N-valued measure on G; we want to define $\Psi(\delta) \in M$. If δ takes values in a finite-dimensional subspace of N, then we can write

$$\delta = \sum_{i=1}^{n} n_i \delta_i,$$

with $n_i \in N$ and δ_i a compactly supported Borel measure on G. I explained that such measures act on representations (like M); so we can define

$$\Psi(\delta) = \sum_{i=1}^{n} \delta_i \cdot \psi(n_i),$$

an element of M. (Very likely there are missing inverses.)

4. Unitary representations. The reason nobody thinks much about these questions is that neither of these two constructions is the most interesting one. That most interesting one, due to Mackey, is *unitary induction*. It's defined for instance whenever G is locally compact and N is a *unitary* representation of the subgroup H (on a Hilbert space). In this case there is a unitary representation $\mathrm{uInd}_H^G(N)$, defined on a certain Hilbert space of maps from G to N which are "square-integrable on the homogeneous space G/H" (a notion which requires a bit of thought to define).

This square-integrability condition is (roughly speaking) weaker (and so defines a larger space) than the continuity condition defining Ind; but it is (roughly speaking) stronger (and so defines a smaller space) than the "measure" condition defining RInd. (Both of these assertions are correct only when G/H is compact. If G/H is noncompact, then a continuous function need not be square-integrable, and a square-integrable function need not define a compactly supported measure.)

A consequence of this intermediate status is that uInd does not satisfy Frobenius reciprocity in either version (left or right adjointness to For). Life is hard.

The key to defining uInd is understanding integration on G/H. There is a more detailed account of this in the notes

`http://math-mit.edu/~dav/integration.pdf`;

here is a summary of what we need. Write δ_G for the (positive-valued continuous character) *modular character* on G. This function is defined by the requirement that if $\mu_{\ell,G}$ is any left Haar measure on G, then

$$\mu_{\ell,G}(Ag) = \delta_G(g)\mu_{\ell,G}(A)$$

for any measurable $A \subset G$. (The reason this function exists is that the right translate by g of the Haar measure $\mu_{\ell,G}$ is another left Haar measure. By the uniqueness of Haar measure, it is a multiple of $\mu_{\ell,G}$.)

Proposition 4.1. *Suppose G is a locally compact group and H is a closed subgroup. Define*

$$\delta_{G/H} \colon H \to \mathbb{R}^{+,\times}, \qquad \delta_{G/H}(h) = \delta_G(h)\delta_H(h^{-1}).$$

Define

$$C_{c,\delta_{G/H}}(G/H) = \{f\colon G \to \mathbb{C} \textit{ cont, cptly supp mod } H \mid f(gh) = \delta_{G/H}(h)^{-1} f(g)\},$$

the space of compactly supported continuous density functions for G/H*. The group* G *acts by left translation on* $C_{c,\delta_{G/H}}(G/H)$*. There is a natural "integral" (uniquely defined up to a positive scalar multiple)*

$$\int_{G/H} f\, d_{G/H} \in \mathbb{C} \qquad (f \in C_{c,\delta_{G/H}}(G/H))$$

that is invariant under left translation, carries positive functions to positive numbers, and in general behaves like a measure.

If G is a Lie group, the space $C_{c,\delta_{G/H}}(G/H)$ may be identified (after choice of a Lebesgue measure on the tangent space at the base point eH) with the space of compactly supported continuous measures on G/H. The "integral" is just the volume of G/H in the measure.

Corollary 4.2 (Mackey). *In the setting of Proposition 4.1, suppose* N *is a unitary representation of the subgroup* H*. Define*

$$(\mathrm{uInd}_H^G)_c(N) = \mathrm{cInd}_H^G(N \otimes \delta_{G/H}^{1/2}),$$

the space of compactly supported continuous functions on G *with values in* N*, transforming on the right by* H *(with the action on* N *twisted by the square root of the positive character* $\delta_{G/H}$*). If* f_1 *and* f_2 *belong to* $(\mathrm{uInd}_H^G)_c(N)$*, then the function on* G

$$\langle f_1, f_2\rangle(x) = \langle f_1(x), f_2(x)\rangle_N$$

belongs to $C_{c,\delta_{G/H}}(G/H)$*. The integral defines a pre-Hilbert space structure*

$$\langle f_1, f_2\rangle = \int_{G/H} \langle f_1(x), f_2(x)\rangle_N \, d_{G/H}(x)$$

on $(\mathrm{uInd}_H^G)_c(N)$*. The action of* G *(by left translation) preserves this pre-Hilbert structure.*

The Hilbert space completion $\mathrm{uInd}_H^G(N)$ *is a (continuous) unitary representation of* G*.*

References

[KV] A. Knapp and D. Vogan, *Cohomological Induction and Unitary Representations*, Princeton University Press, Princeton, New Jersey, 1995.