The size of infinite-dimensional representations

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Introduction

Lessons from finite fields

Lessons from p-adic fields

Repns over $\mathbb{R}$, $\mathbb{C}$

GK dimension and characters

Other real reductive groups

Outline

Introduction

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Repns over $\mathbb{R}$, $\mathbb{C}$

GK dimension and characters

Other real reductive groups

Gelfand's abstract harmonic analysis

Topological grp G acts on X, have questions about X.

Step 1. Attach to X Hilbert space $\mathcal{H}$ (e.g. $L^2(X)$).
Questions about $X \rightsquigarrow$ questions about $\mathcal{H}$.

Step 2. Find finest G-eqvt decomp $\mathcal{H} = \oplus_\alpha \mathcal{H}_\alpha$.
Questions about $\mathcal{H} \rightsquigarrow$ questions about each $\mathcal{H}_\alpha$.

Each $\mathcal{H}_\alpha$ is irreducible unitary representation of G: indecomposable action of G on a Hilbert space.

Step 3. Understand $\widehat{G}_u =$ all irreducible unitary representations of G: unitary dual problem.

Step 4. Answers about irr reps $\rightsquigarrow$ answers about X.

Topic today: what's an irreducible unitary representation look like?

Representations of $GL(V(k))$

Concentrate on group $G(k) = GL(V(k))$ invertible linear transformations of n-diml vector space $V(k)$.

Stay vague about (locally compact) ground field k: mostly $\mathbb{R}$ or $\mathbb{C}$, but $\mathbb{F}_q$, p-adic fields also interesting.

$G(k)$ acts on $(n-1)$-diml (over k) proj alg variety

$$X_{1,n-1}(k) = \{\text{1-diml subspaces of } V(k)\}$$

Hilbert space

$$\mathcal{H}_{1,n-1}(k) = \{L^2 \text{ half-densities on } X_{1,n-1}(k)\}$$

$k = \mathbb{R}, \mathbb{C}$, p-adic: $G(k)$ acts by irr rep $\rho(1, n-1)$.

Question for today: how big is this Hilbert space?

Size of L^2(proj space)

Want "dimension" for inf-diml Hilbert space

$$\mathcal{H}_{1,n-1}(k) = \{L^2 \text{ half-densities on } X_{1,n-1}(k)\}$$

For guidance, look at fin-diml analogue: take base field $k = \mathbb{F}_q$; then $\#V(\mathbb{F}_q) = q^n$,
$G(\mathbb{F}_q) = GL(V(\mathbb{F}_q))$ = finite group of linear transformations of $V(\mathbb{F}_q)$.

$G(\mathbb{F}_q)$ acts on

$$X_{1,n-1}(\mathbb{F}_q) = \{\text{1-diml subspaces of } V(\mathbb{F}_q)\};$$

$$\#X_{1,n-1}(\mathbb{F}_q) = (q^n - 1)/(q-1) = q^{n-1} + q^{n-2} + \cdots + 1.$$

$$\mathcal{H}_{1,n-1}(\mathbb{F}_q) = \{\text{functions on } X_{1,n-1}(\mathbb{F}_q)\}$$

$$\dim \mathcal{H}_{1,n-1}(\mathbb{F}_q) = \#X_{1,n-1}(\mathbb{F}_q) = q^{n-1} + \cdots + 1$$

$$= \text{poly in } q, \text{ degree} = \dim(X_{1,n-1}).$$

About $GL(V(\mathbb{F}_q))$

To understand size of repns of $GL(V)$, need size of $GL(V)$...

The "q-analogue" of m is the polynomial

$$q^{m-1} + q^{m-2} + \cdots + q + 1 = \frac{q^m - 1}{q - 1};$$

value at $q = 1$ is m.

$$(m!)_q = (q^{m-1} + q^{m-2} \cdots + 1)(q^{m-2} + \cdots + 1) \cdots (q+1) \cdot 1$$
$$= \frac{q^m - 1}{q - 1} \cdot \frac{q^{m-1} - 1}{q - 1} \cdots \frac{q^2 - 1}{q - 1} \cdot \frac{q - 1}{q - 1}.$$

(q-analogue of $m!$; poly in q, deg $= \binom{m}{2}$, val at $1 = m!$)

Geometric meaning: number of complete flags in an m-dimensional vector space over $\mathbb{F}_q$.

Cardinality of $GL(V(\mathbb{F}_q))$ is $(n!)_q(q-1)^n q^{\binom{n}{2}}$.

$GL(V(\mathbb{F}_q))$ is "q-analogue" of symmetric group.

More representations over $\mathbb{F}_q$

Continue with $k = \mathbb{F}_q$, $G(\mathbb{F}_q) = GL(V(\mathbb{F}_q))$.

$\pi = (p_1, \ldots, p_m)$, $\sum_j p_j = n$; $G(\mathbb{F}_q)$ acts on

$$X_\pi(\mathbb{F}_q) = \{0 = S_0 \subset S_1 \subset \cdots S_m = V(\mathbb{F}_q), \quad \text{subspace chains, } \dim(S_j/S_{j-1} = p_j\};$$

$\mathbb{F}_q$-variety of dimension

$$d(\pi) =_{\text{def}} \binom{n}{2} - \sum_j \binom{p_j}{2}.$$

$$\#X_\pi(\mathbb{F}_q) = \frac{(n!)_q}{(p_1!)_q (p_2!)_q \cdots (p_m!)_q}.$$

$$\mathcal{H}_\pi(\mathbb{F}_q) = \{\text{functions on } X_\pi(\mathbb{F}_q)\}$$

$$\dim \mathcal{H}_\pi(\mathbb{F}_q) = \#X_\pi(\mathbb{F}_q) = \text{poly in } q \text{ of deg } d(\pi).$$

Repn space $\simeq$ cplx fns on $\mathbb{F}_q$-variety of dim $d(\pi)$

Moral of the $\mathbb{F}_q$ story

$G(\mathbb{F}_q) = GL(V(\mathbb{F}_q)) =$ q-analogue of symm group S_n

irr rep of $G(\mathbb{F}_q) \rightsquigarrow$ partition π of $n \rightsquigarrow X_\pi =$ flags of type π

irr rep $\approx$ functions on $X_\pi(\mathbb{F}_q)$

dim(irr rep) $=$ poly in q of degree $\dim X_\pi$

Problem: what partition is attached to each irr rep?

Dimension of representation provides a clue.

big reps $\leftrightsquigarrow$ partitions with small parts.

Note: partition $\pi \leftrightsquigarrow$ irreducible rep of S_n.

About p-adic $GL(V(k))$

k p-adic field $\supset \mathfrak{O}$ ring of integers $\supset \mathfrak{p}$ maximal ideal

$\mathfrak{O}/\mathfrak{p} = \mathbb{F}_q$ residue field

$V(k)$ n-diml vec space; fix basis $\rightsquigarrow V(k) \simeq k^n$.

Basis $\rightsquigarrow V(\mathfrak{O}) \simeq \mathfrak{O}^n \subset k^n \simeq V(k)$

$G(k) = GL(V(k)) \simeq GL(n,k)$.

For $r \geq 0$, have open subgroups (nbhd base at I)

$$\begin{aligned} G_r &= \{g \in GL(n,\mathfrak{O}) \mid g \equiv I \bmod \mathfrak{p}^r\} \\ &= \text{subgp of } G(\mathfrak{O}) \text{ acting triv on } V(\mathfrak{O})/V(\mathfrak{p}^r)\ . \end{aligned}$$

Note $G_0 = G(\mathfrak{O}) \simeq GL(n,\mathfrak{O})$.

$G_0/G_r \simeq GL(V(\mathfrak{O}/\mathfrak{p}^r))$ finite group, extension of $G(\mathbb{F}_q)$ by nilp gp of order $q^{n^2 r}$.

$G_r \rightsquigarrow$ decompose $G(k)$-spaces, reps.

Flag varieties over p-adic k

$G(k) = GL(V(k)) \supset$ compact open $G_0 \supset G_1 \supset \cdots$

$\pi = (p_1, \ldots, p_m)$, $\sum_j p_j = n$; $G(k)$ acts on

$$X_\pi(k) = \{0 = S_0 \subset S_1 \subset \cdots S_m = V(k), \quad \text{subspace chains, } \dim(S_j/S_{j-1} = p_j\}$$

$$\downarrow \simeq$$

$$X_\pi(\mathfrak{O}) = \{0 = L_0 \subset L_1 \subset \cdots L_m = V(\mathfrak{O}), \quad \text{lattice chains, } \mathrm{rk}(L_j/L_{j-1}) = p_j\}$$

$$\downarrow \pi_r$$

$$X_\pi(\mathfrak{O}/\mathfrak{p}^r) = \{0 = \ell_0 \subset \ell_1 \subset \cdots \ell_m = V(\mathfrak{O}/\mathfrak{p}^r), \quad \text{submodule chains, } \mathrm{rk}(\ell_j/\ell_{j-1}) = p_j\}$$

π_r fibers $= G_r$ orbits on $X_\pi(k)$; number of orbits is

$$\#X_\pi(\mathfrak{O}/\mathfrak{p}^r) = \frac{(n!)_q}{(p_1!)_q(p_2!)_q\cdots(p_m!)_q} \cdot q^{rd(\pi)}.$$

Some representations over p-adic k

$G(k) = GL(V(k)) \supset$ (small) compact open G_r

$\pi = (p_1, \ldots, p_m)$, $\sum_j p_j = n$; $G(k)$ acts on

$$X_\pi(k) = \text{subspace chains of type } \pi$$

$$d(\pi) =_{\text{def}} \binom{n}{2} - \sum_m \binom{p_m}{2} = \dim X_\pi$$

Hilbert space

$$\mathcal{H}_\pi(k) = \{L^2 \text{ half-densities on } X_\pi(k)\}$$

carries unitary rep $\rho(\pi)$ of $G(k)$; space is incr union

$$\mathcal{H}_\pi(k)^{G_0} \subset \mathcal{H}_\pi(k)^{G_1} \subset \cdots \subset \mathcal{H}_\pi(k)^{G_r} \subset \cdots$$

finite-diml reps of G_0.

$$\begin{aligned}\dim(\mathcal{H}_\pi(k)^{G_r}) &= \text{number of orbits of } G_r \text{ on } X_\pi(k)\\ &= \frac{(n!)_q}{(p_1!)_q(p_2!)_q\cdots(p_m!)_q} \cdot q^{r\,d(\pi)}.\end{aligned}$$

General representations over p-adic k

$$\pi = (p_1, \ldots, p_m), \quad \sum_j p_j = n$$

$$X_\pi(k) = \text{subspace chains of type } \pi$$

$$\mathcal{H}_\pi(k) = \{L^2 \text{ half-densities on } X_\pi(k)\}$$

$$\dim(\mathcal{H}_\pi(k)^{G_r}) = \frac{(n!)_q}{(p_1!)_q (p_2!)_q \cdots (p_m!)_q} \cdot q^{rd(\pi)}$$

Theorem (Shalika germs)

If $(\rho, \mathcal{H})$ arb irr rep of $G(k)$, then for every partition π of n there is an integer $a_\pi(\rho)$ so that for $r \geq r(\rho)$

$$\mathcal{H} \simeq \sum_\pi a_\pi \mathcal{H}_\pi(k)$$

as (virtual) representations of G_r.

Corollary

$$\dim \mathcal{H}^{G_r} = \textit{poly in } q^r \textit{ of deg } d(\pi(\rho)),$$

some partition $\pi(\rho)$, and all $r \geq r(\rho)$.

Moral of the p-adic story

$G(k) = GL(V(k))$ has neighborhood base at 1 of compact open subgroups $G_0 \supset G_1 \supset \cdots \supset G_r \supset \cdots$

irr rep of $G(k) \rightsquigarrow$ partition $\pi(\rho)$ of $n \rightsquigarrow X_\pi =$ flags of type π

irr rep on $\mathcal{H} \approx$ functions on $X_\pi(k)$

$\dim(\mathcal{H}^{G_r}) =$ poly in q^r of deg $d(\pi) = \dim X_\pi$ (large r)

Problem: what partition is attached to each irr rep?

Rate of growth of chain of subspaces

$$\mathcal{H}_\pi^{G_0} \subset \mathcal{H}_\pi^{G_1} \subset \cdots \mathcal{H}_\pi^{G_r} \subset \cdots$$

provides a clue.

big reps $\leftrightsquigarrow$ partitions with small parts.

Representations of $GL(V(\mathbb{R}))$

$G(\mathbb{R}) = GL(V(\mathbb{R})) \simeq GL(n, \mathbb{R})$.

$G(\mathbb{R})$ acts on $(n-1)$-diml compact manifold

$$X_{1,n-1}(\mathbb{R}) = \{\text{1-diml subspaces of } V(\mathbb{R})\} \simeq \mathbb{RP}^{n-1}$$

$$\mathcal{H}_{1,n-1}(\mathbb{R}) = \{L^2 \text{ half-densities on } \mathbb{RP}^{n-1}\}$$

Hilbert space carrying irr unitary rep of $G(\mathbb{R})$.

Question for today: how big is this Hilbert space? Can we extract $n-1$ from it?

Difficulty: all inf-diml separable Hilbert spaces are isomorphic (as Hilbert spaces).

Same problem for other function spaces:

$$C^\infty(\mathbb{RP}^{n-1}) \simeq C^\infty(\mathbb{RP}^{m-1}) \text{ as topological vec space}$$

$$\mathbb{C}[x_1, \ldots, x_{n-1}] \simeq \mathbb{C}[y_1, \ldots, y_{m-1}] \text{ as vec space}$$

Distinguish using exhaustion by fin-diml subspaces.

Lessons from real analysis

X compact d-diml Riemannian, Δ_X Laplacian

$\mathcal{H} = L^2(X)$, $\mathcal{H}_\lambda = \lambda$-eigenspace of Δ_X.

Theorem (Weyl)

If $\mathcal{H}(N) = \sum_{\lambda \le N^2} \mathcal{H}_\lambda$, *then* $\dim \mathcal{H}(N) \sim c_X N^d$.

Conclude: $\dim X \leftrightsquigarrow$ asymp distn of Δ_X eigenvalues

Example: $X = \mathbb{RP}^{n-1}$, $C^\infty(X) =$ homog even fns on $\mathbb{R}^n$.

$\mathcal{H}_{2k(2k+(n-1))} \simeq$ deg $2k$ pols mod $r^2 \cdot$ (deg $2(k-1)$ pols)

$$\dim \mathcal{H}_{2k(2k+(n-1))} = \frac{[(2k+1)(2k+2)\cdots(2k+n-3)][4k+n-2]}{(n-2)!},$$

polynomial in k of degree $n-2$.

$$\mathcal{H}\left(2k\sqrt{1+\frac{n-1}{2k}}\right) \simeq S^{2k}(\mathbb{R}^n)$$

$$\dim \mathcal{H}\left(2k\sqrt{1+\frac{n-1}{2k}}\right) = \binom{n+2k-1}{n-1},$$

polynomial in k of degree $n-1$.

$O(n) \subset GL(n, \mathbb{R})$ commutes with Δ_X, preserves $\mathcal{H}_\lambda$.

More representations over $\mathbb{R}$

Choice of basis defines compact subgroup

$$O(n) \subset G(\mathbb{R}) = GL(V(\mathbb{R})) \simeq GL(n, \mathbb{R}).$$

Casimir $\Omega_{O(n)} = -\sum X_i^2$, $\{X_i\}$ orth basis of Lie $O(n)$.

$\pi = (p_1, \ldots, p_m)$, $\sum_j p_j = n$; $G(\mathbb{R})$ acts on cpt Riemannian

$$X_\pi(\mathbb{R}) = \text{subspace chains of type } \pi$$

$$d(\pi) =_{\text{def}} \binom{n}{2} - \sum_m \binom{p_m}{2} = \dim X_\pi$$

$O(n)$ transitive on $X_\pi(\mathbb{R})$, $\Delta_{X_\pi} =$ action of $\Omega_{O(n)}$; isotropy

$$O(\pi) =_{def} O(p_1) \times \cdots O(p_m) \subset O(n).$$

Unitary rep $\rho(\pi)$ on $\mathcal{H}_\pi(\mathbb{R}) = L^2(X_\pi(\mathbb{R}))$; res to $O(n)$ is

$$\mathrm{Ind}_{O(\pi)}^{O(n)}(\mathbb{C}) = \sum_{\mu \in \widehat{O(n)}} (\dim \mu^{O(\pi)})\mu$$

Therefore compute Laplacian eigenvalue distribution

$$\mathcal{H}_\pi(N) = \sum_{\mu(\Omega) \leq N^2} (\dim \mu^{O(\pi)})\mu.$$

$\dim \mathcal{H}_\pi(N) \sim a(\pi) N^{d(\pi)}$: res to $O(n)$ computes $d(\pi)$.

General representations over $\mathbb{R}$

$(\rho, \mathcal{H})$ arbitrary irr rep of $G(\mathbb{R}) \simeq GL(n, \mathbb{R})$.
Restriction to cpt subgp $O(n)$ decomposes

$$\mathcal{H} \simeq \sum_{\mu \in \widehat{O(n)}} m_\rho(\mu)\mu \qquad (m_\rho(\mu) \text{ non-neg integer}).$$

Example of $\mathcal{H}_\pi = L^2(X_\pi)$ suggests defining

$$\mathcal{H}(N) =_{\text{def}} \sum_{\mu(\Omega) \le N^2} m_\rho(\mu)\mu.$$

Theorem

There is partition $\pi(\rho)$ of n, pos integer $c(\rho)$ so that

$$\dim \mathcal{H}(N) \sim c(\rho) a(\pi(\rho)) N^{d(\pi(\rho))}.$$

Recall that $\dim \mathcal{H}_\pi(N) \sim a(\pi) N^{d(\pi)}$.

Definition

For ρ irr rep of $G(\mathbb{R})$, the Gelfand-Kirillov dimension of ρ is the non-neg integer $\text{Dim}(\rho) = d(\pi(\rho))$; measures asymp distn of eigenvalues of Casimir $\Omega_{O(n)}$ in ρ.

(First) moral of the real story

$G(\mathbb{R}) = GL(V(\mathbb{R}))$ has compact subgroup $O(n)$.

irr rep of $G(\mathbb{R}) \rightsquigarrow$ partition $\pi(\rho)$ of $n \rightsquigarrow X_\pi =$ flags of type π

irr rep on $\mathcal{H} \approx$ functions on $X_\pi(\mathbb{R})$, cpt homog space for $G(\mathbb{R})$ and for $O(n)$. Precisely:

asymp distn of eigenvalues of Casimir $\Omega_{O(n)}$ in $\rho \rightsquigarrow$ eigenvals of Laplacian on $X_\pi(\mathbb{R})$.

Problems: what partition is attached to each irr rep?

what else does partition tell you about irr rep?

To address these questions, use characters of reps. . .

Distribution characters

Idea of Gelfand-Kirillov dimension began with *dimension* for fin-diml irr rep $(\rho, \mathcal{H})$ of G.

Can write $\dim \rho = \operatorname{tr} \mathrm{Id}_{\mathcal{H}} = \operatorname{tr} \rho(1)$.

Useful to consider character of ρ, function on G:

$$\Theta_\rho(g) =_{\mathrm{def}} \operatorname{tr} \rho(g),$$

because character of ρ determines ρ up to equiv.

Inf-diml irr $(\rho, \mathcal{H})$: $\rho(g)$ never trace class. *Regularize*...

$G(\mathbb{R}) = GL(V(\mathbb{R}))$, δ cptly supp test density on $G(\mathbb{R})$,

$$\rho(\delta) = \int_{G(\mathbb{R})} \rho(g)\delta(g) \in \mathrm{End}(\mathcal{H})$$

is trace class operator (Harish-Chandra).

Map $\Theta_\rho(\delta) = \operatorname{tr} \rho(\delta)$ is generalized function on $G(\mathbb{R})$.

GK dim of $\rho \leftrightsquigarrow$ singularity of Θ_ρ at $1 \in G(\mathbb{R})$.

More lessons from real analysis

f smooth on vec space $W(\mathbb{R})$, $f_t(w) = f(tw)$; $\overset{\text{Taylor}}{\rightsquigarrow}$
$f_t \sim \sum_{k=0}^{\infty} t^k P_k, \qquad (t \to \infty)$, P_k homog deg k poly.

Seek analogous expansion for non-smooth gen fns.

Theorem (Barbasch-V)

Θ_ρ distn char of irr rep ρ of $G(\mathbb{R})$, $\overset{\text{exp}}{\rightsquigarrow}$ gen fn θ_ρ on

$$\mathfrak{g}(\mathbb{R}) = \text{Lie}(G(\mathbb{R})) = n \times n \text{ real matrices}$$

Then θ_ρ has asymptotic expansion

$$\theta_{\rho,t} \sim \sum_{k=-d(\rho)}^{\infty} t^k T_k(\rho),$$

$T_k(\rho)$ tempered gen fn homog of deg k.

Leading terms match: $T_{-d(\rho)}(\rho) = c(\rho) T_{-d(\pi)}(\rho(\pi(\rho)))$.

Conclusion: char Θ_ρ near $1 \in G(\mathbb{R})$ equal to $c(\rho) \cdot \Theta_{\rho(\pi)}$ modulo lower order terms.

More about character leading terms

Looked at expansion $\theta_{\rho,t} \sim \sum_{k=-d(\rho)}^{\infty} t^k T_k(\rho)$.

Fin-diml rep: $d(\rho) = 0$, leading term $T_0(\rho) = \dim \rho$.

Leading term $T_{-d(\rho)} \leftrightsquigarrow$ analogue of dimension

Example: $G(\mathbb{R})$ action on $X_\pi(\mathbb{R}) \rightsquigarrow$ moment map

$$\mu_\pi \colon T^* X_\pi(\mathbb{R}) \to \mathfrak{g}(\mathbb{R})^*.$$

μ_π is birational onto closure of nilpotent conj class

$$\mathcal{O}_{\pi^t} \subset \mathfrak{g}(\mathbb{R})^* \simeq n \times n \text{ real matrices;}$$

Natural measure on $T^* X_\pi(\mathbb{R}) \overset{\mu_\pi}{\rightsquigarrow}$ measure on $\mathcal{O}_{\pi^t}$ $\overset{\text{Fourier}}{\rightsquigarrow}$ generalized function on $\mathfrak{g}(\mathbb{R})$.

Leading term $T_{-d(\pi)}(\rho(\pi))$ is Fourier transform $\widehat{\mathcal{O}_{\pi^t}}$.

(Second) moral of the real story

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Other real reductive groups

$G(\mathbb{R}) = GL(V(\mathbb{R}))$

irr rep ρ of $G(\mathbb{R})$

$\xrightarrow{\text{trace}}$ distribution character Θ_ρ (gen fn on $G(\mathbb{R})$)

$\xrightarrow{\text{exp}}$ generalized function θ_ρ on $\mathfrak{g}(\mathbb{R})$

$\xrightarrow{\text{asymp expansion}}$ $T_{-d(\rho)}(\rho)$ temp, deg $-d(\rho)$ gen fn on $\mathfrak{g}(\mathbb{R})$

$\xrightarrow{\text{Fourier}}$ tempered degree $[-\dim(\mathfrak{g}(\mathbb{R})) + d(\rho)]$ distribution on $\mathfrak{g}(\mathbb{R})^* \simeq n \times n$ real matrices

$\xrightarrow{\text{support}}$ conjugacy class $\mathcal{O}_{\pi^t}$ of real nilp matrices

$\xrightarrow{\text{Jordan}}$ partition $\pi(\rho)$ of n

That finds the partition attached to each irr rep.

Other real reductive groups

$G(\mathbb{R})$ real reductive group, $K(\mathbb{R})$ maximal compact subgroup, $\Omega_{K(\mathbb{R})}$ Casimir operator for $K(\mathbb{R})$.

Example: $Sp(2n, \mathbb{R})$, $\mathbb{R}$-linear transf of $\mathbb{C}^n$ preserving symplectic form

$$\omega(v, w) = \mathrm{Im}\langle v, w\rangle$$

(imag part of std Herm form); $K(\mathbb{R}) = U(n)$.

Example: $O(p, q)$ linear transf of $\mathbb{R} \times \mathbb{R}^q$ preserving symmetric form

$$\langle(v_1, v_2), (w_1, w_2)\rangle_{p,q} = \langle v_1, w_1\rangle - \langle v_2, w_2\rangle;$$

$K(\mathbb{R}) = O(p) \times O(q)$.

(Al)most general example: $G(\mathbb{R}) \subset GL(N, \mathbb{R})$ closed subgp preserved by transpose, $K(\mathbb{R}) = G(\mathbb{R}) \cap O(N)$.

Big idea:

$$G(\mathbb{R}) \text{ rep “size”} \leftrightsquigarrow \text{restriction to } K(\mathbb{R}) \text{ asymptotics}$$

GK dimension for other real reductive

$G(\mathbb{R})$ real reductive group, $K(\mathbb{R})$ maximal compact subgroup, $\Omega_{K(\mathbb{R})}$ Casimir operator for $K(\mathbb{R})$.

$(\rho, \mathcal{H})$ irr rep of $G(\mathbb{R})$; then (Harish-Chandra)

$$\mathcal{H} \simeq \sum_{\mu \in \widehat{K(\mathbb{R})}} m_\rho(\mu)\mu, \qquad (m_\rho(\mu) \text{ non-neg integer}).$$

As for $GL(n)$, can define

$$\mathcal{H}(N) =_{\text{def}} \sum_{\mu(\Omega_{K(\mathbb{R})}) \leq N^2} m_\rho(\mu)\mu.$$

Theorem

There is a non-negative integer $d(\rho)$ and a positive constant $b(\rho)$ so that

$$\dim \mathcal{H}(N) \sim b(\rho) N^{d(\rho)}.$$

Call $d(\rho)$ the Gelfand-Kirillov dimension of ρ.

What's wrong with GK dimension for other G

Case of $GL(n)$: have special homog spaces $X_\pi(\mathbb{R})$ (partial flag variety) so that reps $L^2(X_\pi(\mathbb{R}))$ "approximately model" any irr rep.

Other $G(\mathbb{R})$: have analogues of X_π (real flag varieties); but they no longer model *all* irr reps.

Example: $G(\mathbb{R}) = Mp(4, \mathbb{R})$ nonlinear double cover of symplectic group. Four possible spaces "X_π":

point $X_\emptyset$ (dim $= 0$)
(isotropic) lines $X_1 = \{L_1 \subset \mathbb{R}^4\} = \mathbb{RP}^3$ (dim 3)
Lagrangian planes $X_2 = \{L_2 \subset \mathbb{R}^4\} \simeq U(2)/O(2)$ (dim 3)
isotr. flags $X_{12} = \{L_1 \subset L_2 \subset \mathbb{R}^4\} \simeq U(2)/O(1) \times O(1)$ (dim 4)

Get GK dims 0, 3, 4; metaplectic repn has GK dim 2.

But asymptotic expansion of characters still works...

Character expansions for real groups

$G(\mathbb{R})$ real reductive group, $(\rho, \mathbb{H})$ irr rep

δ cptly supp test density on $G(\mathbb{R}) \rightsquigarrow$ trace class op

$$\rho(\delta) = \int_{G(\mathbb{R})} \rho(g)\delta(g) \in \mathrm{End}(\mathcal{H})$$

Map $\Theta_\rho(\delta) = \mathrm{tr}\,\rho(\delta)$ is generalized function on $G(\mathbb{R})$.

Lift via exp to gen fn θ_ρ on $\mathfrak{g}(\mathbb{R}) = \mathrm{Lie}(G(\mathbb{R}))$

Theorem (Barbasch-V)

θ_ρ has asymptotic expansion $\theta_{\rho,t} \sim \sum_{k=-d(\rho)}^{\infty} t^k T_k(\rho)$, $T_k(\rho)$ tempered gen fn homog of deg k.
Leading term $T_{-d(\rho)}$ is finite linear comb of Fourier transforms of invt measures on nilp orbits in $\mathfrak{g}(\mathbb{R})^*$:

$$T_{-d(\rho)} = \sum_{\dim \mathcal{O} = 2d(\rho)} c(\rho, \mathcal{O})\widehat{\mathcal{O}}.$$

(Schmid-Vilonen) Coeffs $c(\rho, \mathcal{O})$ are non-neg ints.

(Third) moral of the real story

$G(\mathbb{R})$ real reductive

irr rep ρ of $G(\mathbb{R})$

$\xrightarrow{\text{trace...support}}$ non-neg integer comb

$$T_{-d(\rho)} = \sum_{\dim \mathcal{O}=2d(\rho)} c(\rho, \mathcal{O})\widehat{\mathcal{O}}.$$

of several nilpotent orbits of $G(\mathbb{R})$ on $\mathfrak{g}(\mathbb{R})^*$

More to do...

Can (approx) describe $\rho|_{K(\mathbb{R})}$ with orbits $\mathcal{O}$.

Relate unitarity of ρ to expansion; not understood.

Seek to compute constants $c(\rho, \mathcal{O})$ using KL calculation of character Θ_ρ; not understood.