

# GENERAL PARABOLIC SUBGROUPS OF $GL_n(\mathbb{C})$

MATTHEW GARCIA

## 1. PARABOLIC SUBGROUPS

In this lecture we will describe the general form of the parabolic subgroups of  $GL_n(\mathbb{C})$ . Before we go on to discuss the specific case of  $GL_n(\mathbb{C})$  we describe parabolic subgroups of any group. To begin we refresh our memories on solvable groups. We can find the derived series of a group  $G$  by iteratively finding the commutator subgroup. Recall that the commutator subgroup is defined as

$$G' = \langle xyx^{-1}y^{-1} | x, y \in G \rangle.$$

The group  $G$  is called *solvable* if the derived series eventually terminates to the trivial group. That is to say,  $G$  is solvable if and only if all subgroups from the series defined by

$$G^{[n]} = \langle xyx^{-1}y^{-1} | x, y \in G^{[n-1]} \rangle$$

are equal the trivial group for some integer  $n$  and integers larger than that  $n$ . We now define Borel subgroup.

**Definition 1.1.** *A subgroup  $B$  of a group  $G$  is called Borel if it is a maximal solvable connected algebraic subgroup of  $G$ .*

In this context we take maximal to mean that a Borel subgroup is not contained in any other solvable subgroup of  $G$ . The words connected and algebraic have to do with properties of groups that would divert our purpose too far to discuss in detail here. Let us just note that most complex linear groups are both algebraic and connected. Conveniently, all Borel subgroups are conjugate to one and other. Therefore, in computation we may work with just one Borel subgroup and deduce results for any Borel subgroup. The Borel subgroup is intimately related to the notion of a parabolic subgroup.

**Definition 1.2.** *A subgroup  $P$  of a group  $G$  is called a parabolic subgroup if it properly contains a Borel subgroup  $B$  of  $G$ .*

With these few abstract notions it is hard to tell what step to take next. Nevertheless, we may now set  $G = GL_n(\mathbb{C})$  and describe the general form of that group's parabolic subgroups.

## 2. GENERAL LINEAR GROUP OVER THE COMPLEX FIELD

Our task now is to determine one of the Borel subgroups of  $GL_n(\mathbb{C})$ . We wish to show that all Borel subgroups of  $GL_n(\mathbb{C})$  are conjugate to the subgroup  $B$  of invertible  $n \times n$  upper triangular matrices over  $\mathbb{C}$ . To verify this we must show two things: that  $B$  is solvable, and that no solvable subgroup of  $GL_n(\mathbb{C})$  properly contains  $B$ .

To show that  $B$  is solvable we apply the method of Lie algebras. We state here without proof that  $B$  is a connected Lie group. We do remark, however, that the connectedness of  $B$  relies on the fact that we are working over the complex field. The Lie algebra of  $B$  is the set of all  $n \times n$  upper triangular matrices, denoted  $\mathfrak{B}$ . Because  $B$  is connected, we can conclude that  $B$  is solvable if  $\mathfrak{B}$  is solvable. For  $\mathfrak{B}$  to be solvable the Lie commutator series

$$\mathfrak{B}^1 = [\mathfrak{B}, \mathfrak{B}], \mathfrak{B}^2 = [\mathfrak{B}^1, \mathfrak{B}^1], \dots$$

must eventually vanish,  $\mathfrak{B}^i = 0$  for some  $i$ . Note that the above notation is set theoretical. Here the bracket is defined

$$[X, Y] = XY - YX$$

for  $X, Y \in \mathfrak{B}$ . Take arbitrary  $M, N \in \mathfrak{B}$ .  $[M, N] = P$  is an arbitrary element of  $\mathfrak{B}^1$  and has the form

$$P = \begin{pmatrix} 0 & p_{12} & p_{13} & \dots & p_{1n} \\ 0 & 0 & p_{23} & \dots & p_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

All matrices in  $\mathfrak{B}^1$  have this form. If we continue computing the Lie commutator series with each iterated application of the bracket twice the number of left most nonzero terms in each row that went to zero in the last iteration become zero. Therefore, after a finite number,  $k$ , of recursive applications of the Lie bracket we find that  $\mathfrak{B}^k = 0$  and we can conclude that  $B$  is a solvable Lie group.

To prove that no solvable subgroup of  $GL_n(\mathbb{C})$  properly contains  $B$  we show that no subgroup containing  $B$  is solvable. Assume that  $P$  is some subgroup of  $GL_n(\mathbb{C})$  satisfying  $B \subset P \subset GL_n(\mathbb{C})$ . Then we can deduce that an element  $A \in P$  has the block form

$$A = \begin{pmatrix} A_1 & * & * & \dots & * \\ 0 & A_2 & * & \dots & * \\ 0 & 0 & A_3 & \dots & * \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & A_r \end{pmatrix},$$

where each of the  $A$ -blocks has size  $a_i \times a_i$ .  $P$  consists of all such block matrices.

Now we can construct the group homomorphism

$$\phi : P \rightarrow GL_{a_1}(\mathbb{C}) \times \dots \times GL_{a_r}(\mathbb{C})$$

defined by taking the diagonal blocks of an element  $A$  of  $P$  and listing them as an ordered  $r$ -tuple of matrices. By the definition of the kernel of a homomorphism  $\ker \phi$  is normal in  $P$ . We can, therefore, form the quotient group  $\frac{P}{\ker \phi}$ . We know that if  $P$  is solvable then  $\frac{P}{\ker \phi}$  is also solvable. Hence, if the quotient group is not solvable then, by the contrapositive,  $P$  is not solvable. The quotient group  $\frac{P}{\ker \phi}$  is isomorphic to the direct product  $GL_{a_1}(\mathbb{C}) \times \dots \times GL_{a_r}(\mathbb{C})$ . This cross product is clearly not solvable unless  $a_i = 1$  for all  $i$ , since  $GL_n(\mathbb{C})$  is not solvable for any  $n$ . However, if  $a_i = 1$  for all  $i$  then  $P \cong B$ . Therefore,  $B$  is maximal and finally we can conclude that  $B$  is a Borel subgroup of  $GL_n(\mathbb{C})$ . Any parabolic subgroup of  $GL_n(\mathbb{C})$  properly contains  $B$  as defined above.