Free associative algebras

February 16, 2015

The point of these notes is to recall some linear algebra that we'll be using in many forms in 18.745. You can think of the notes as a makeup for the canceled class February 10.

Vector spaces can be thought of as a very nice place to study addition. The notion of direct sum of vector spaces provides a place to add things even in different vector spaces, and turns out to be a very powerful tool for studying vector spaces. The theory of bases says that any vector space can be written as a direct sum of lines.

In the same way, algebras are a very nice place to study multiplication. There is a notion analogous to direct sum, called *tensor product*, which makes it possible to multiply things even in different vector spaces. The theory of tensor products turns out to be a very powerful tool for studying algebras. I won't write down the definition of tensor product (which you can find in lots of serious algebra books, or even on Wikipedia); but I will write the key property relating them to bilinear maps.

Suppose U_1, U_2, and W are vector spaces over the same field k. A *bilinear map* from $U_1 \times U_2$ to W is a function

$$\beta\colon U_1 \times U_2 \to W \tag{0.1}$$

which is assumed to be linear in each variable separately:

$$\begin{aligned} \beta(au_1 + bu_1', u_2) &= a\beta(u_1, u_2) + b\beta(u_1', u_2), \\ \beta(u_1, cu_2 + u_2') &= c\beta(u_1, u_2) + d\beta(u_1, u_2') \end{aligned} \tag{0.2}$$

(all u_1, u_1' in U_1, all u_2, u_2' in U_2, and all a, b, c, d in k). Examples of bilinear maps that you should know include a bilinear form on a vector space V_0 (in which case U_1 and U_2 are both V_0, and W is k); and composition of linear maps (in which case U_1 is $\mathrm{Hom}(V_2, V_3)$; U_2 is $\mathrm{Hom}(V_1, V_2)$; and W is $\mathrm{Hom}(V_1, V_3)$). (I am writing $\mathrm{Hom}(V, W)$ for the vector space of all k-linear maps from V to W.)

You should convince yourself that that the collection of bilinear maps from $U_1 \times U_2$ to W is in a natural way a vector space. As far as I can tell there is not a standard notation for this vector space, but we could call it

$$\mathrm{Bil}(U_1, U_2; W) = \text{all bilinear maps from } U_1 \times U_2 \text{ to } W. \tag{0.3}$$

You should convince yourself that there are natural vector space isomorphisms

$$\mathrm{Bil}(U_1, V_2; W) \simeq \mathrm{Hom}(U_1, \mathrm{Hom}(U_2, W)) \simeq \mathrm{Hom}(U_2, \mathrm{Hom}(U_1, W)). \tag{0.4}$$

Here are the basic properties of tensor products of vector spaces. Suppose U_1 and U_2 are vector spaces over the same field k. The *tensor product* of U_1 and U_2 is a vector space $U_1 \otimes U_2$ over k with the following properties

1. There is a bilinear map
$$U_1 \times U_2 \to U_1 \otimes U_2, \qquad (u_1, u_2) \mapsto u_1 \otimes u_2. \tag{0.5a}$$

2. Tensor product is a covariant functor in the variables U_1 and U_2. That is, any linear maps $T_i \colon U_i \to V_i$ induce a linear map
$$T_1 \otimes T_2 \colon U_1 \otimes U_2 \to V_1 \otimes V_2, \qquad T_1 \otimes T_2(u_1 \otimes u_2) = (T_1(u_1)) \otimes (T_2(u_2)). \tag{0.5b}$$

3. If $\{u_1^i \mid i \in I\}$ is a basis of U_1, and $\{u_2^j \mid j \in J\}$ is a basis of U_2, then $\{u_1^i \otimes u_2^j \mid (i, j) \in I \times J\}$ is a basis of $U_1 \otimes U_2$.

4. If U_1 and U_2 are finite-dimensional, then
$$\dim(U_1 \otimes U_2) = \dim(U_1) \cdot \dim(U_2). \tag{0.5c}$$

5. If $\beta \colon U_1 \times U_2 \to W$ is any bilinear map, then there is a unique linear map $B \colon U_1 \otimes U_2 \to W$ with the property that
$$B(u_1 \otimes u_2) = \beta(u_1, u_2) \qquad (u_1 \in U_1,\ u_2 \in U_2) \tag{0.5d}$$

The last statement is the *universality* property which is the reason-for-being of the tensor product. It may be written

$$\mathrm{Bil}(U_1, U_2; W) \simeq \mathrm{Hom}(U_1 \otimes U_2, W). \tag{0.5e}$$

In terms of the natural isomorphisms (0.4), this amounts to

$$\operatorname{Hom}(U_1 \otimes U_2, W) \simeq \operatorname{Hom}(U_1, \operatorname{Hom}(U_2, W)) \simeq \operatorname{Hom}(U_2, \operatorname{Hom}(U_1, W)). \tag{0.5f}$$

The first equality says that the tensor product functor $U_1 \mapsto U_1 \otimes U_2$ is a left adjoint to the Hom functor $W \mapsto \operatorname{Hom}(U_2, W)$. (In both cases U_2 is a fixed "parameter.")

Everything that's written above about "bi" can be replaced by "multi," if we start not with two vector spaces U_1 and U_2 but with n (for any positive integer n). We can therefore define an *n-fold tensor product* $U_1 \otimes \cdots \otimes U_n$, spanned by elements $u_1 \otimes \cdots \otimes u_n$, with the property that

$$\operatorname{Mult}(U_1, \ldots, U_n; W) \simeq \operatorname{Hom}(U_1 \otimes \cdots \otimes U_n, W). \tag{0.6a}$$

Here's why it's not necessary to define n-fold tensor products separately. Suppose $n = p + q$, with p and q both positive integers. It's easy to see that an n-linear map from $U_1 \times \cdots \times U_n$ to W is the same thing as a bilinear map

$$\operatorname{Mult}(U_1, \ldots, U_p; W) \times \operatorname{Mult}(U_{p+1}, \ldots, U_{p+q}; W). \tag{0.6b}$$

This statement amounts to a natural isomorphism

$$[U_1 \otimes \cdots \otimes U_p] \otimes [U_{p+1} \otimes \cdots U_{p+q}] \simeq U_1 \otimes \cdots U_n. \tag{0.6c}$$

Repeating this procedure, we find that *any n-fold tensor product can be expressed as an iterated binary tensor product,* in fact usually in several different ways. For example,

$$[U_1 \otimes U_2] \otimes [U_3] \simeq U_1 \otimes U_2 \otimes U_3 \simeq [U_1] \otimes [U_2 \otimes U_3]. \tag{0.6d}$$

The number of different ways of doing this in general is the $(n+1)$st Catalan number: the number of ways of writing an n-fold product as an iteration of 2-fold products. Each way corresponds to some distribution of parentheses in the iterated product; the isomorphisms arise by removing parentheses or putting them back in. A little more precisely, the isomorphisms in (0.6d) are characterized by

$$[u_1 \otimes u_2] \otimes [u_3] \mapsto u_1 \otimes u_2 \otimes u_3 \mapsto [u_1] \otimes [u_2 \otimes u_3].$$

Recall that an *algebra* over a field k is a vector space A over k endowed with a *bilinear* multiplication map

$$*\colon A \times A \to A. \tag{0.7a}$$

Equivalently (because of (0.5d)), we can think of $*$ as a *linear* map

$$\Pi\colon A \otimes A \to A. \tag{0.7b}$$

(The capital Π is meant to stand for "product.") Because of functoriality of tensor product (0.5b), linear maps

$$i_V\colon V \to A, \qquad i_W\colon W \to A$$

induce

$$i_{V\otimes W} = \Pi \circ (i_V \otimes i_W)\colon V \otimes W \to A.$$

We say that V *generates* A if the smallest subalgebra of A containing the image of i_V is A itself. This smallest subalgebra is written

$$\langle V\rangle = \text{smallest subalgebra containing } \operatorname{im} i_V. \tag{0.7c}$$

It is easy to see that

$$\langle V\rangle = \operatorname{im} i_V + \operatorname{im} i_{V\otimes V} + \operatorname{im} i_{[V\otimes V]\otimes[V]} + \operatorname{im} i_{[V]\otimes[V\otimes V]} + \cdots \tag{0.7d}$$

Here the first term is the image of V; the second is the span of all products of two elements of V; and the next two are all products of three elements of V. The next terms in $\cdots$ are five terms like $i_{[V\otimes V]\otimes[V\otimes V]}$ representing all products of four terms from V (with parentheses arranged in any of the five possible ways).

The kernels of these linear maps can be thought of as *relations*: special properties of the multiplication in A that are not consequences of the bilinearity assumption. For example, the assertion that elements of V satisfy the commutative law amounts to the statement that

$$\ker i_{V\otimes V} \supset \{v\otimes v' - v'\otimes v \mid v, v' \in V\}.$$

The assertion that elements of V satisfy the associative law is

$$i_{[V\otimes V]\otimes[V]} = i_{[V]\otimes[V\otimes V]};$$

here the equality is interpreted by identifying the domains of the linear maps using (0.6d).

To say that a mathematical object is "free" is to say that it the only relations that are true are the ones forced by the nature of the object. This leads to the following two definitions.

Definition 0.7e. Suppose V is a vector space over k. The *free algebra generated by* V is the direct sum

$$\mathcal{F}(V) = V \oplus (V \otimes V) \oplus ([V \otimes V] \otimes V) \oplus (V \otimes [V \otimes V]) \oplus \cdots$$

The summands are various iterated tensor products of V; there is one iterated tensor product with n terms for each way of writing the n-fold tensor product as an iteration of 2-fold tensor products (that is, the $n-1$st Catalan nunber).

The definition of multiplication $*$ on $\mathcal{F}(V)$ is that a p-fold iterated tensor product times a q-fold iterated tensor product is given by the corresponding $p+q$-fold iterated product. That is, for example,

$$\star\colon V \times [V \otimes V] \otimes V \to V \otimes [[V \otimes V] \otimes V]$$

is the natural map of (0.5a). In words, $\mathfrak{F}(V)$ is spanned by all possible iterated tensor products of elements of V:

$$v_1,\ v_2 \otimes v_3,\ [v_4 \otimes v_5] \otimes v_6,\ v_7 \otimes [v_8 \otimes v_9],$$

and so on.

If we impose the associative law, we get a much smaller and more comprehensible algebra.

Definition 0.7f. Suppose V is a vector space over k. For $k \geq 1$, the kth tensor power of V is the k-fold tensor power

$$T^k(V) = V \otimes \cdots \otimes V \qquad (k \text{ factors}).$$

We define $T^0(V) = k$. The *free associative algebra generated by* V, more often called the *tensor algebra of* V, is

$$T(V) = \bigoplus_{k=0}^{\infty} T^k(V).$$

The algebra structure, usually written $\otimes$, is given by the identifications (0.6c) of iterated tensor products with n-fold tensor products:

$$\otimes\colon T^p(V) \times T^q(V) \to T^{p+q}(V);$$

I'll leave to you the little modifications needed for p or q equal to zero.

The tensor algebra is actually the free associative algebra with (multiplicative) identity. To get a free associative algebra without identity, we should take just $\bigoplus_{k=1}^{\infty} T^k(V)$; but we'll have no occasion to do that.

The definition of the Weyl algebra in the second problem set is

$$A(V) = T(V)/(\text{ideal generated by } v \otimes w - w \otimes v - \omega(v, w)); \qquad (0.8)$$

here v and w are elements of V, so $v \otimes w - w \otimes v \in T^2(V)$, and $\omega(v, w)$ is an element of $k = T^0(V)$.