3. GEOMETRY OF FLAG MANIFOLDS AND REPRESENTATION THEORY.

These notes are intended to be a substitute for the cancelled class on March 19. The title is more appropriate for a multi-volume book series than for a few pages, but (as they say in the shooting-into-the-air business (see

http://www.theonion.com/archive/archive_39.html

issue 06)) it's important to aim high.

A possible subtitle for the book series would be, "Why representation theorists and algebraic geometers don't understand each other." One of the fundamental meta-facts about general algebraic varieties is that they have very few subvarieties. Faltings' work in the direction of Fermat's last theorem says that for $n \geq 3$, the projective algebraic curve $x^n + y^n + z^n = 0$ has only a finite number of rational points. Representation theory people tend to be interested mostly in a very small family of very peculiar algebraic varieties, which are overflowing with subvarieties. This difference in perspective shows up at the very beginning, with projective space. For algebraic geometers, projective space is just some large boring workshop in which one can fashion beautiful little pieces of art. For representation theorists, projective space is (more or less) the only algebraic variety you need.

Here is the plan. I'll sketch the easiest facts about flag manifolds; then I'll explain how to interpret them as facts about homogeneous spaces for $GL(n)$ (modulo parabolic subgroups). In this form they make sense for arbitrary reductive groups (modulo parabolic subgroups), and I'll state the corresponding facts there. Finally I'll try to relate this geometry to the reducibility of principal series representations.

Much of this material makes sense over arbitrary base fields (instead of the real numbers), and I'll try to keep that possibility in view. The representation theory at the end needs some topology on the groups, but still makes sense over any locally compact field.

So suppose F is any field and V is an n-dimensional vector space over F. The *projective space of* V is

$$\begin{aligned} \mathbb{P}(V) &= \text{one-dimensional subspaces of } V \\ &\simeq \{V - \{0\}\}/F^\times. \end{aligned} \tag{3.1}$$

A more algebro-geometric statement would be that these are the F-rational points of the algebraic variety consisting of one-dimensional $\overline{F}$-subspaces of the vector space $V(\overline{F}) = V \otimes_F \overline{F}$. The fact that any line defined over F actually has non-zero points in V—that is, non-trivial points defined over F—is either almost trivial (from the point of view of linear algebra), or deeply surprising (from the point of view of algebraic geometry).

The variety $\mathbb{P}(V)$ carries some interesting vector bundles. (A vector bundle means at least an F-vector space attached to each point of $\mathbb{P}(V)$. Since I have

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not introduced any topology, we can't speak about "continuous dependence" of the vector space on the point. In this generality the best way to impose such continuity is using algebraic geometry; but since I have already adopted a scornfully negative tone toward algebraic geometry, I will not pursue it. You can read about algebraic vector bundles in Hartshorne.) First there is the *tautological line bundle* $\mathcal{L}$, whose fiber $\mathcal{L}_x$ at a point $x \in \mathbb{P}(V)$ is x, regarded as a one-dimensional subspace of V. Next, there is the trivial n-dimensional vector bundle $\mathcal{V}$, whose fiber at every point x is equal to V. (The total space of $\mathcal{V}$ is therefore $\mathbb{P}(V) \times V$.) We can combine one interesting example and one trivial one to get another interesting example, the $(n-1)$-dimensional *quotient vector bundle* $\mathcal{Q}$, whose fiber $\mathcal{Q}_x$ at x is $V/\mathcal{L}_x$. Using these building blocks, one can construct more vector bundles using the vector space functors of linear algebra: dual space, tensor powers, exterior powers, and so on.

It is worth pausing a moment to consider the question of finding sections of these vector bundles. (Recall that a section of a vector bundle $\mathcal{V} \to X$ is a map from X to $\mathcal{V}$ whose value at x belongs to the vector space $\mathcal{V}_x$.) A section of the tautological line bundle $\mathcal{L}$ on $\mathcal{P}(V)$ assigns to each line in V a particular element of that line. If V is one-dimensional (so that $\mathcal{P}(V)$ consists of a single point), then a section of $\mathcal{L}$ is just an element of V. If the dimension of V is greater than one, then the only "nice" way (precisely, the only algebraic way) to pick a point in every line is to pick zero in every line. That is, the only (algebraic) section of $\mathcal{L}$ is zero.

An (algebraic) section of $\mathcal{V}$ is an (algebraic) map from the (connected projective) variety $\mathbb{P}(V)$ to the (affine) variety V. Such a map must be constant: the only (algebraic) sections of $\mathcal{V}$ are the constant sections, sending each line x to the same vector $v \in V$.

A section of $\mathcal{Q}$ assigns to each point x a class in the quotient space $V/\mathcal{L}_x$. If V has dimension one, the quotient bundle is zero, and obviously the only section is zero. If V has dimension greater than one, then (not so obviously) the algebraic sections of $\mathcal{Q}$ are all given by fixing $v \in V$, and assigning to x the class

$$\xi_v(x) = x + \mathcal{L}_x \in \mathcal{Q}_x.$$

In this case the section ξ_v vanishes precisely on the set of lines containing v. This zero set is all of $\mathbb{P}(V)$ if $v = 0$, and it is the single point $Fv \in \mathbb{P}(V)$ if $v \neq 0$.

I said that $\mathbb{P}(V)$ was the only algebraic variety we would need. This was a pedagogically useful half-truth. The next order of business is to introduce partial flag varieties (a necessary generalization of projective space) and then to see in what sense they are built out of projective spaces. A *flag size for* n is a subset π of $\{0, 1, 2, \ldots, n-1, n\}$ that contains both 0 and n. We write it as

$$\pi = \{p_0, \ldots, p_m\}, \qquad 0 = p_0 < p_1 < \cdots < p_{m-1} < p_m = n. \tag{3.2(a)}$$

A *flag of size* π *in* V is a collection x of subspaces of V,

$$0 = W_0(x) \subset W_1(x) \subset \cdots \subset W_{m-1}(x) \subset W_m(x) = V, \tag{3.2(b)}$$

subject to the conditions

$$\dim W_i(x) = p_i. \tag{3.2(c)}$$

We are now in a position to define the *variety of partial flags in* V *of size* π as

$$\begin{aligned} \mathbb{F}_\pi(V) &= \text{flags of size } \pi \text{ in } V \\ &= \{\text{collections } x = \{W_i(x)\} \text{ of subspaces of } V \text{ with } \dim W_i(x) = p_i, \\ &\qquad 0 = W_0(x) \subset W_1(x) \subset \cdots \subset W_{m-1}(x) \subset W_m(x) = V\}. \end{aligned}$$

Just as for projective space, we could make a more algebro-geometric formulation: that these are the F-rational points of the algebraic variety consisting of flags of size π in the $\overline{F}$-vector space $V(\overline{F})$.

Some of these partial flag varieties are very familiar. If $\pi = \{0, n\}$, then the unique flag of size π is $0 \subset V$; so $\mathbb{F}_{\{0,n\}}(V)$ consists of a single point. If $\pi = \{0, 1, n\}$ (and n is at least 1, so that π is a well-defined flag size for n) then $\mathbb{F}_\pi(V) = \mathbb{P}(V)$. More generally, if k is any integer between 0 and n, and $\pi = \{0, k, n\}$ then $\mathbb{F}_\pi(V) = \mathrm{Gr}_k(V)$, the Grassmann variety of k-dimensional subspaces of V.

Each partial flag variety comes equipped with many natural vector bundles. Still writing π as in (3.2)(a), we will be interested particularly in the bundles $\mathcal{E}(\pi)_i$ (for $i = 1, \dots, m$) defined by

$$\mathcal{E}(\pi)_{i,x} = W_i(x)/W_{i-1}(x) \tag{3.3}$$

for $x \in \mathbb{F}_\pi(V)$ and with notation as in (3.2)(b). This is a vector bundle of dimension $p_i - p_{i-1}$.

Here is a collection of elementary facts about partial flag varieties.

Theorem 3.4. *Suppose that V is an n-dimensional vector space over F, and $\pi \subset \rho$ are two flag sizes for n (cf. (3.2)). Write*

$$\pi = \{p_0, p_1, \dots, p_m\}, \qquad 0 = p_0 < p_1 < \cdots < p_{m-1} < p_m = n.$$

The larger flag size $\rho = \{r_0, \dots, r_M\}$ is obtained by interspersing additional terms among the p's. List the terms added between p_{i-1} and p_i as

$$p_{i-1} = p_{i-1} + a_0^i < p_{i-1} + a_1^i < p_{i-1} + a_2^i < \cdots p_{i-1} + a_{M(i)}^i = p_i.$$

Then

$$\{r_j \mid 0 \le j \le M\} = \{p_{i-1} + a_k^i \mid 1 \le i \le m, \quad 0 \le j \le M(i)\}.$$

(1) *The appearance of the p_i among the r_j may be written explicitly as*

$$p_0 = r_0,\ p_1 = r_{M(1)},\ p_2 = r_{M(1)+M(2)},\ \dots,\ p_m = r_{M(1)+\cdots+M(m)} = n.$$

Furthermore

$$\alpha^i = \{a_0^i, \dots, a_{M(i)}^i\}$$

is a flag size for $p_i - p_{i-1}$.

(2) *There is a natural map (actually a morphism of algebraic varieties)*

$$\phi_\pi^\rho \colon \mathbb{F}_\rho(V) \to \mathbb{F}_\pi(V),$$

sending the point $y = \{W_j(y)\}$ to

$$x = \phi_\pi^\rho(y), \qquad W_i(x) = W_{M(1)+M(2)+\cdots+M(i)}(y).$$

(3) *The fiber of ϕ_π^ρ over the point $x \in \mathbb{F}_\pi(V)$ may be identified with a product of partial flag varieties. Explicitly,*

$$(\phi_\pi^\rho)^{-1}(x) \simeq \mathbb{F}_{\alpha_1}(W_1(x)/W_0(x)) \times \cdots \times \mathbb{F}_{\alpha_m}(W_m(x)/W_{m-1}(x)).$$

(4) *Let* $\mathcal{E}(\pi)_i$ *be the vector bundle over* $\mathbb{F}_\pi(V)$ *defined in (3.3), and let*

$$\mathcal{F} = (\phi_\pi^\rho)^* \mathcal{E}(\pi)_i$$

be its pullback to $\mathbb{F}_\rho(V)$. *Then* $\mathcal{F}$ *has a filtration by subbundles*

$$0 = \mathcal{F}^0 \subset \mathcal{F}^1 \subset \cdots \subset \mathcal{F}^{M(i)} = \mathcal{F},$$

with the property that

$$\mathcal{F}^j/\mathcal{F}^{j-1} \simeq \mathcal{E}(\rho)_{M(1)+\cdots+M(i-1)+j}.$$

Sketch of proof. Part (1) is just a matter of sorting out the notation. The map in part (2) is well-defined because of (1). Part (3) follows from the identification of subspaces of $W_i(x)$ containing $W_{i-1}(x)$ with subspaces of $W_i/W_{i-1}(x)$. In part (4), the fiber of $\mathcal{F}$ at y is by definition the fiber of $\mathcal{E}_i$ at $x = \phi_\pi^\rho(y)$. This fiber is

$$W_i(x)/W_{i-1}(x) = W_{\sum_{k=1}^{i} M(k)}(y)/W_{\sum_{k=1}^{i-1} M(k)}(y).$$

This fiber is filtered by the various $W_j(y)$, with j running from $\sum_{k=1}^{i-1} M(k)$ to $\sum_{k=1}^{i} M(k)$. The statement follows. □

Suppose $\mathcal{V} \to X$ is an n-dimensional vector bundle over a space X, and π is a flag size for n. We can make a new space $\mathbb{F}_\pi(\mathcal{V}) \to X$: the fiber over a point $x \in X$ is $\mathbb{F}_\pi(\mathcal{V}_x)$, the collection of all flags of size π in the vector space $\mathcal{V}_x$. If we are working in the category of smooth manifolds, then $\mathbb{F}_\pi(\mathcal{V})$ is a smooth manifold, and the map to X is a proper submersion. In the category of algebraic varieties, the map from $\mathbb{F}_\pi(\mathcal{V})$ to X is again smooth and proper, so that the new variety is smooth whenever X is. In either of these settings we can describe the tangent space to $\mathbb{F}_\pi(\mathcal{V})$ at a point ξ that maps to $x \in X$: there is a short exact sequence of vector spaces

$$0 \to T_\xi(\mathbb{F}_\pi(\mathcal{V}_x)) \to T_\xi(\mathbb{F}_\pi(\mathcal{V})) \to T_x(X) \to 0. \tag{3.5}$$

Here the first term is a tangent space to a flag variety, and the last is a tangent space to the base space X.

The simplest case of this construction has $\pi = \{0, k, n\}$; then $\mathbb{F}_\pi(\mathcal{V})$ is the bundle of k-dimensional subspaces of $\mathcal{V}$. When $k = 1$ we get $\mathbb{P}(\mathcal{V})$, the bundle of lines in the vector bundle $\mathcal{V}$. In differential geometry it is perhaps more common to work with the bundle of spheres in a vector bundle with a metric, but the idea is exactly the same.

Theorem 3.4 exhibits the flag variety $\mathbb{F}_\rho(V)$ (corresponding to the larger flag size ρ) as a such a flag bundle over $\mathbb{F}_\pi(V)$. (More precisely, the construction in the theorem involves several different vector bundles.) If we obtain ρ from π by adding a single dimension $p_{i-1} + k$ between p_{i-1} and p_i, then $\mathbb{F}_\rho(V)$ is the Grassmanian bundle of k-dimensional subspaces of the vector bundle $\mathcal{E}(\pi)_i$:

$$\mathbb{F}_\rho(V) = \mathbb{F}_{\{0,k,p_i - p_{i-1}\}}(\mathcal{E}(\pi)_i). \tag{3.6}$$

By iterating this construction (adding numbers to a flag size one by one), we find that any partial flag variety is a bundle of Grassmannian varieties over a bundle of Grassmannian varieties over ... .

So what? One consequence is that information about Grassmannian varieties can often be inflated into information about general flag varieties. Here is an example.

Theorem 3.7. *Suppose that V is an n-dimensional vector space over F, and ρ is a flag size for n (cf. (3.2)). Write*

$$\rho = \{r_0, r_1, \ldots, r_M\}, \qquad 0 = r_0 < r_1 < \cdots < r_{M-1} < r_M = n.$$

(1) *The dimension of $\mathbb{F}_\pi(V)$ (as an algebraic variety, or as a smooth manifold if $F = \mathbb{R}$) is equal to*

$$\sum_{1 \le i < j \le M} (r_j - r_{j-1})(r_i - r_{i-1}) = \sum_{j=1}^{M} (r_j - r_{j-1}) r_{j-1} = \sum_{i=1}^{M} (n - r_i)(r_i - r_{i-1}).$$

(2) *The tangent bundle $T\mathbb{F}_\pi(V)$ has a filtration with subquotients*

$$\mathrm{Hom}(\mathcal{E}(\rho)_i, \mathcal{E}(\rho)_j), \qquad 0 < i < j \le m.$$

Here $\mathcal{E}(\rho)_i$ is the vector bundle defined in (3.3).

Sketch of proof. The tangent bundle may be interpreted in the sense of smooth manifolds if $F = \mathbb{R}$; for other fields one should use a definition from algebraic geometry. The calculation of dimensions in (1) is immediate from the statement about tangent bundles in (2), so we concentrate on that. The proof will proceed by induction on M. If $M = 1$, then $\rho = \{0, n\}$, and the flag variety is reduced to a single point. The tangent bundle is zero, which has a trivial filtration with no subquotients, as is asserted in (2). So we may assume that $M \ge 2$. In this case we can find a flag size π for n by removing exactly one element of ρ (other than 0 or n). We are then in the setting of Theorem 3.4, with $m = M - 1$; so we fix a point y of $\mathbb{F}_\rho(V)$ mapping to x. The various flag sizes α_i defined in Theorem 3.4 are all the trivial ones $\{0, p_i - p_{i-1}\}$, with a single exception

$$\alpha_{i_0} = \{0, k, p_{i_0} - p_{i_0 - 1}\}. \tag{3.8(a)}$$

The flag varieties $\mathbb{F}_{\alpha_i}(W_i(x)/W_{i-1}(x))$ are therefore all just single points, except for

$$\mathbb{F}_{\alpha_{i_0}}(W_{i_0}(x)/W_{i_0-1}(x)) = k\text{-dimensional subspaces of } W_{i_0}(x)/W_{i_0-1}(x) \tag{3.8(b)}$$

As a consequence of part (3) of Theorem 3.4, and of (3.5), we get a short exact sequence

$$0 \to T_y(\mathbb{F}_{\alpha_{i_0}}(W_{i_0}(x)/W_{i_0-1}(x)) \to T_y(\mathbb{F}_\rho(V)) \to T_x(\mathbb{F}_\pi(V)) \to 0. \tag{3.8(c)}$$

By inductive hypothesis, the last tangent space $T_x\mathbb{F}_\pi$ has a filtration with subquotients equal to the various

$$\mathrm{Hom}(\mathcal{E}(\pi)_{i,x}, \mathcal{E}(\pi)_{j,x}), \qquad 1 \le i < j \le M - 1. \tag{3.8(d)}$$

Because of the relationship between π and ρ, and the definition (3.3), we find

$$\mathcal{E}(\pi)_{i,x} = \mathcal{E}(\rho)_{i,y} \qquad 1 \le i < i_0; \tag{3.9(a)}$$

there is a short exact sequence

$$0 \to \mathcal{E}(\rho)_{i_0,y} \to \mathcal{E}(\pi)_{i_0,x} \to \mathcal{E}(\rho)_{i_0+1,y} \to 0; \qquad (3.9)(b)$$

and

$$\mathcal{E}(\pi)_{i,x} = \mathcal{E}(\rho)_{i+1,y} \qquad i_0 < i < M. \qquad (3.9)(c)$$

Consequently

$$\mathrm{Hom}(\mathcal{E}(\pi)_{i,x}, \mathcal{E}(\pi)_{j,x}) = \mathrm{Hom}(\mathcal{E}(\rho)_{i,y}, \mathcal{E}(\rho)_{j,y}), \qquad 1 \leq i < j < i_0; \qquad (3.9)(d)$$

$$\mathrm{Hom}(\mathcal{E}(\pi)_{i,x}, \mathcal{E}(\pi)_{j,x}) = \mathrm{Hom}(\mathcal{E}(\rho)_{i,y}, \mathcal{E}(\rho)_{j+1,y}), \qquad 1 \leq i < i_0 < j < M; \qquad (3.9)(e)$$

and

$$\mathrm{Hom}(\mathcal{E}(\pi)_{i,x}, \mathcal{E}(\pi)_{j,x}) = \mathrm{Hom}(\mathcal{E}(\rho)_{i+1,y}, \mathcal{E}(\rho)_{j+1,y}), \qquad i_0 < i < j < M. \quad (3.9)(f)$$

Similarly, there are short exact sequences

$$\begin{gathered} 0 \to \mathrm{Hom}(\mathcal{E}(\rho)_{i,y}, \mathcal{E}(\rho)_{i_0,y}) \to \mathrm{Hom}(\mathcal{E}(\pi)_{i,x}, \mathcal{E}(\pi)_{i_0,x}) \to \\ \mathrm{Hom}(\mathcal{E}(\rho)_{i,y}, \mathcal{E}(\rho)_{i_0+1,y}) \to 0, \qquad 1 \leq i < i_0, \end{gathered} \qquad (3.9)(g)$$

and

$$\begin{gathered} 0 \to \mathrm{Hom}(\mathcal{E}(\rho)_{i_0+1,y}, \mathcal{E}(\rho)_{j+1,y}) \to \mathrm{Hom}(\mathcal{E}(\pi)_{i_0,x}, \mathcal{E}(\pi)_{j,x}) \to \\ \mathrm{Hom}(\mathcal{E}(\rho)_{i_0,y}, \mathcal{E}(\rho)_{j+1,y}) \to 0, \qquad i_0 < j < M. \end{gathered} \qquad (3.9)(h)$$

Combining the various identifications in (3.9) with (3.8)(d), we find that the tangent space $T_x\mathbb{F}_\pi(V)$ has a filtration with subquotients equal to all of the various

$$\mathrm{Hom}(\mathcal{E}(\rho)_{i,y}, \mathcal{E}(\rho)_{j,y}), \qquad 1 < i < j \leq M$$

except for $i = i_0$ and $j = i_0 + 1$. In light of (3.8)(c), we will therefore be done as soon as we prove

Lemma 3.10. *Suppose V is an n-dimensional vector space, and W is a subspace of dimension k. Write $\alpha = \{0, k, n\}$ for the flag size corresponding to the Grassmannian of k-dimensional subspaces. Identify W with a point y of $\mathbb{F}_\alpha(V)$. Then the tangent space at y to the Grassmannian is*

$$T_y(\mathbb{F}_\alpha(V)) = \mathrm{Hom}(W, V/W) = \mathrm{Hom}(\mathcal{E}(\alpha)_{1,y}, \mathcal{E}(\alpha)_{2,y}).$$

This is the case $M = 2$ of Theorem 3.7.

Proof of Lemma 3.10. We use the group $G = GL(V)$ of invertible linear transformations on V. We can take $F = \mathbb{R}$ and think of G as a Lie group, or work over an arbitrary field and think of G as an algebraic group. In either case G is an open subset of the vector space $\mathrm{End}\, V$ of all linear transformations of V. Consequently the Lie algebra $\mathfrak{g}$ of G, which as a vector space is the tangent space at the identity to G, may be identified naturally with $\operatorname{End} V$. The Lie bracket is commutator of linear transformations. If G acts transitively on a manifold (or algebraic variety) X, then the isotropy group $H = G_x$ of a point $x \in X$ is a Lie (or algebraic) subgroup of G. Under a separability assumption (automatic if F has characteristic zero, and true for the Grassmannian variety in general; I won't discuss it, but you could look at [Borel], Proposition II.6.7) there is a natural identification of tangent spaces

$$T_x(X) \simeq T_{eH}(G/H) \simeq \mathfrak{g}/\mathfrak{h}. \tag{3.11(a)}$$

The action of $GL(V)$ on k-dimensional subspaces of V is transitive, so (writing H for the stabilizer of the subspace W) we get $\mathbb{F}_\alpha(V) \simeq G/H$. Clearly

$$H = \{g \in GL(V) \mid g(W) \subset W\} = \{g \in G \mid \text{image of } g(W) \text{ in } V/W \text{ is zero}\}. \tag{3.11(b)}$$

This condition is easy to differentiate, and we find that the Lie algebra of H is

$$\mathfrak{h} = \{T \in \operatorname{End} V \mid T(W) \subset W\} = \{T \in \operatorname{End} V \mid \text{ image of } T(W) \text{ in } V/W \text{ is zero}\}. \tag{3.11(c)}$$

Dividing all endomorphisms by this subspace, we find

$$\mathfrak{g}/\mathfrak{h} \simeq \operatorname{Hom}(W, V/W). \tag{3.11(d)}$$

□

Since we had already succeeded in reducing Theorem 3.7 to the case of Grassmann varieties, this completes the proof of Theorem 3.7. □

References

[Borel] A. Borel, *Linear Algebraic Groups*, Springer-Verlag, New York, 1991.