Errata for "Parameters for twisted representations"

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1 Introduction

{sec:intro}

The article [1] describes an algorithm for computing the unitary dual of a real reductive algebraic group $G(\mathbb{R})$. One ingredient in the algorithm is the Kazhdan-Lusztig polynomials defined and computed in [4]. These polynomials are indexed by pairs (J, J') of irreducible representations of $G(\mathbb{R})$. {ALTV} {LVold}

A second ingredient in the unitarity algorithm is a twisted version of these polynomials introduced in [5]. The setting involves an outer automorphism δ of $G(\mathbb{R})$ of order two, and the corresponding extended group ${}^\delta G(\mathbb{R})$ (containing $G(\mathbb{R})$ as a subgroup of index two). These twisted polynomials are indexed by pairs $(\tilde{J}, \tilde{J}')$ of extensions to ${}^\delta G(\mathbb{R})$ of irreducible representations of $G(\mathbb{R})$. Each δ-fixed irreducible J of $G(\mathbb{R})$ admits exactly two extensions $\tilde{J}_{+1}$ and $\tilde{J}_{-1}$ to ${}^\delta G(\mathbb{R})$. Roughly speaking, the twisted polynomials depend only on the underlying $G(\mathbb{R})$ representations. Precisely, if $\tilde{J}_{\pm 1}$ are the two extensions of a $G(\mathbb{R})$ irreducible J, and $\tilde{J}'_{\pm 1}$ the two extensions of J', then {LVnew}

$$P_{\tilde{J}_\epsilon, \tilde{J}'_\phi} = \epsilon\phi P_{\tilde{J}_1, \tilde{J}'_1}.$$

The difficulty is that (despite the misleading notation $\tilde{J}_{\pm 1}$) there is no *preferred* extension of J to ${}^\delta G(\mathbb{R})$. A representation like J can be specified precisely using (any of various versions of) a *Langlands parameter* p. The point of the paper [2] was to introduce *extended parameters* E ([2, Definition 5.4]). An extended parameter consists of a Langlands parameter p and some additional {twisted}

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data (for which there are up to equivalence exactly two choices). The Langlands parameter specifies an irreducible $J(p)$ for $G(\mathbb{R})$. The equivalence class of E specifies precisely one extension $\tilde{J}(E)$ to ${}^\delta G(\mathbb{R})$.

Given this precise specification of extended group representations, the algorithm of [5] could be formulated in terms of extended parameters E. This {LVnew} formulation was also presented in [2], and it is there that (at least one) error {twisted} arose.

Here is the nature of the error. The algorithms of [5] involve various linear {LVnew} maps T_κ defined on $\mathbb{Z}[q]$-linear combinations of extended group representations. These formal linear combinations are subject to the relations

$$\tilde{J}_{+1} = -\tilde{J}_{-1}.$$

A typical step in the algorithm involves two to four representations J_i and says something like this: extensions $\tilde{J}_i$ of J_i may be chosen so that

$$T_\kappa(\tilde{J}_1) = \tilde{J}_1 + \tilde{J}_3 + \tilde{J}_4, \quad T_\kappa(\tilde{J}_2) = \tilde{J}_2 + \tilde{J}_3 - \tilde{J}_4 \tag{1.1}$$ {e:LVnew}

(see [5, (7.6i'')]). *If one replaces any* $\tilde{J}_i$ *by the* other *extension of* J_i*, then the* {LVnew} *sign of the coefficent of the* $\tilde{J}_i$ *term in each such formula must change.*

For each of the cases considered in [5], there is an explanation in [2] of how {LVnew}{twisted} to choose extended parameters so that the formulas in [5] are true. The error {LVnew} is that *for the case* `2i12` *described in* [2, Lemma 8.1]*, the choices are incorrect.* {twisted} More precisely, the formulas [2, (44)] must be replaced by {twisted}

$$\begin{aligned}
T_\kappa(E_0) &= E_0 + F_0 + (-1)^{\langle\sigma,t\rangle} F_0' \\
T_\kappa(E_0') &= E_0' + F_0 + (-1)^{\langle\sigma,t\rangle} F_0' \\
T_\kappa(F_0) &= (q^2-1)(E_0 + E_0') + (q^2-2)F_0 \\
T_\kappa(F_0') &= (-1)^{\langle\sigma,t\rangle}(q^2-1)(E_0 - E_0') + (q^2-2)F_0'.
\end{aligned} \tag{1.2}$$ {e:erratum}

(What has been added is the factors $(-1)^{\langle\sigma,t\rangle}$.) We will sketch a proof of these corrected formulas in Section 2. For the introduction, we will say a word about the source of the error. All of the formulas in [5] concern behavior of {LVnew} sheaves on G (or rather on some version of G defined over a finite field) in the direction of some very small Levi subgroup L of G: the group L is locally isomorphic to $SL(2)$, $SL(2)\times SL(2)$, or $SL(3)$, in each case times a torus factor. Standard techniques allow one to prove the formulas working in L rather than in G; so one is ultimately making statements about the representation theory of $L(\mathbb{R})$. Standard techniques very often allow one to reduce representation-theory questions about reductive groups to the case of semisimple groups, since the center necessarily acts by scalars in an irreducible representation. This technique was used (correctly) in [5] to prove (1.1). It was used sloppily to {LVnew} justify [2, Lemma 8.1]. The Lemma is true when G is locally isomorphic to {twisted} $SL(2) \times SL(2)$; but the definitions around extended parameters allow what happens on the center to affect signs. The result is that one can construct extended parameters for a group locally isomorphic to $SL(2) \times SL(2) \times \mathbb{C}^\times$ for which [2, Lemma 8.1] fails. {twisted}

One might hope that therefore the result is true for semisimple G, but this also fails: this bad $SL(2) \times SL(2) \times \mathbb{C}^\times$ example turns up inside $SO(p,q)$.

Now that we have your attention, we will conclude this introduction with a much more ordinary error: the first formula

$$\mathrm{sgn}(E,E') = i^{\langle ({}^\vee\delta_0 - 1)\lambda, t'-t\rangle + \langle \tau'-\tau, (\delta_0-1)\ell'\rangle} (-1)^{\langle \tau, \ell'-\ell\rangle + \langle \lambda'-\lambda, t'\rangle + \langle \tau, t'-t\rangle} \tag{1.3}$$ {e:6.5badsgn}

from [2, Proposition 6.5] is incorrect: the plus sign between the two terms in the exponent of i should be a minus. The corrected formula is {twisted}

$$\mathrm{sgn}(E,E') = i^{\langle ({}^\vee\delta_0 - 1)\lambda, t'-t\rangle - \langle \tau'-\tau, (\delta_0-1)\ell'\rangle} (-1)^{\langle \tau, \ell'-\ell\rangle + \langle \lambda'-\lambda, t'\rangle + \langle \tau, t'-t\rangle}. \tag{1.4}$$ {e:6.5goodsgn}

2 Two copies of $SL(2)$

{sec:twoSL2}

Here is a corrected replacement of [2, Lemma 8.2]. The hypotheses are somewhat different (roughly speaking, more general) from those of the original; after sketching a proof, we will see how this corrected statement leads to (1.2). Notation is as in [2]. {twisted} {twisted} {lemma:8.1corr}

Lemma 2.1. *Suppose κ is of type* `2i12f` *for $E = (\lambda, \tau, \ell, t)$. Define*

$$\ell^{split} = \ell + [(g_\alpha - \ell_\alpha - 1)/2]\alpha^\vee + [(g_\beta - \ell_\beta - 1)/2]\beta^\vee.$$

Suppose that

$$F = (\lambda', \tau', \ell^{split}, t)$$

is an extended parameter of type `2r21f` *appearing in $T_\kappa(E)$. Then the coefficient with which it appears is the ratio of the z-values for these two extended parameters (see* [2, Definition 5.5]*). Explicitly, this is* {twisted}

$$-z(\lambda', \tau', \ell^{split}, t)/z(\lambda, \tau, \ell, t) = i^{\langle \tau', (\delta-1)\ell^{split}\rangle - \langle \tau, (\delta-1)\ell\rangle} (-1)^{\langle \lambda'-\lambda, t\rangle}.$$

{se:twoSL2}

Proof. As mentioned in the introduction, the definition of T_κ involves sheaves on a form of G defined over a finite field. One can make the computation entirely in the Levi subgroup of G defined by

$$\kappa = (\alpha, \beta) = (\alpha, {}^\vee\delta(\alpha)). \tag{2.2a}$$

We may therefore assume that G is equal to L. Writing Z for the identity component of the center of G, this means that

$$G \text{ is a quotient of } SL(2) \times SL(2) \times Z \tag{2.2b}$$

by a finite central subgroup; the first $SL(2)$ corresponds to α and the second to β. So there is a natural identification of Lie algebras

$$\mathfrak{g} = \mathfrak{sl}(2) \times \mathfrak{sl}(2) \times \mathfrak{z}. \tag{2.2c}$$

We use the standard torus

$$H = \left\{ \left[\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}, \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix}, z \right] \mid x, y \in \mathbb{C}^\times, z \in Z \right\} \\ = \left\{ (x, y, z) \mid x, y \in \mathbb{C}^\times, z \in Z \right\}. \tag{2.2d}$$

(Note that H is a *quotient* of $\mathbb{C}^\times \times \mathbb{C}^\times \times Z$, not a direct product.) The Lie algebra of H is identified in this way as

$$\mathfrak{h} \simeq \mathbb{C} \times \mathbb{C} \times \mathfrak{z}, \qquad L \mapsto (\alpha(L)/2,\ \beta(L)/2,\ L_Z) = (L_\alpha/2,\ L_\beta/2,\ L_Z); \tag{2.2e}$$ {e:hcoord}

here L_Z is the projection of L on $\mathfrak{z}$. The simple coroots are

$$H_\alpha = \left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, 0 \right] = (1, 0, 0) \\ H_\beta = \left[\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, 0 \right] = (0, 1, 0). \tag{2.2f}$$

The pinning is given by the simple root vectors

$$X_\alpha = \left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, 0 \right] \\ X_\beta = \left[\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, 0 \right]. \tag{2.2g}$$ {e:roots}

The Tits group generators are

$$\sigma_\alpha = \left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, 1 \right] \\ \sigma_\beta = \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 1 \right]. \tag{2.2h}$$ {e:tits}

{se:strat}

Here is the strategy of the proof. The terms ℓ and ℓ^{split} in our extended parameters define strong involutions ξ and ξ^{split}, and therefore subgroups

$$K_\xi = G^\xi, \qquad K_{\xi^{\text{split}}} = G^{\xi^{\text{split}}}. \tag{2.3a}$$

These have index two in the corresponding subgroups of the extended group

$$^{\delta_0}K_\xi = [^{\delta_0}G]^\xi, \qquad ^{\delta_0}K_{\xi^{\text{split}}} = [^{\delta_0}G]^{\xi^{\text{split}}}. \tag{2.3b}$$ {e:extK}

The hypothesis that F appears in $T_\kappa(E)$ means in particular that ξ^{split} is conjugate to ξ by a unique coset gK_ξ.

The extended parameters E and F define

$$J(E) = \text{irreducible } (\mathfrak{g}, {}^{\delta_0}K_\xi)\text{-module} \\ I(F) = \text{standard } (\mathfrak{g}, {}^{\delta_0}K_{\xi^{\text{split}}})\text{-module} \\ I(F)^{\text{new}} = \text{standard } (\mathfrak{g}, {}^{\delta_0}K_\xi)\text{-module}; \tag{2.3c}$$

the last is obtained by twisting $I(F)$ by $\mathrm{Ad}(g)$.

So what is the representation-theoretic interpretation of the coefficient of F in $T_\kappa(E)$? The multiplicity matrix m (giving multiplicities of irreducibles J as composition factors of standard modules I) is essentially defined by

$$I = \sum_{J \text{ irreducible}} m(J,I)J. \tag{2.3d}$$ {e:multform}

The inverse matrix M writes an irreducible representation J' as an integer combinations of standard representations I':

$$J' = \sum_{I' \text{ standard}} M(I',J')I'. \tag{2.3e}$$ {e:charform}

That the matrices m and M are inverses is more or less a definition.

Suppose now that E and F are representation parameters differing by a single link, which is an ascent from E to F. The entries indexed by (E,F) are just one off the diagonal of these upper triangular unipotent matrices; so the inverse relationship gives

$$m(J(E), I(\pm F)^{\text{new}}) = -M(I(E), J(\pm F)^{\text{new}}). \tag{2.3f}$$ {e:linkinverse}

The Kazhdan-Lusztig polynomials actually compute dimensions of stalks of some perverse cohomology sheaves, and the character formulas (2.3e) involve those dimensions with a $(-1)^{\text{codimension}}$ factor. The conclusion is that

$$M(I(E), J(F)^{\text{new}}) - M(I(E), J(-F)^{\text{new}}) = (-1)^{l(F)-l(E)} P^{\text{tw}}_{E,F}(1). \tag{2.3g}$$ {e:KLchar}

Here $I(-F)^{\text{new}}$ means $I(F)^{\text{new}}$ tensored with the nontrivial character of ${}^{\delta_0}G/G$, the other extension of the standard representation to the extended group.

The (twisted) Kazhdan-Lusztig algorithm in our case says that

$$P^{\text{tw}}_{E,F} = \text{coeff. of } F \text{ in } T_\kappa(E). \tag{2.3h}$$ {e:Tkappachar}

Combining the last three equations gives

$$\begin{aligned} \text{coeff. of } F \text{ in } T_\kappa(E) = -(-1)^{l(F)-l(E)} \\ \left[m(J(E), I(F)^{\text{new}}) - m(J(E), I(-F)^{\text{new}})\right]. \end{aligned} \tag{2.3i}$$

In our present case of length difference 2, this is

$$\text{coeff. of } F \text{ in } T_\kappa(E) = -m(J(E), I(F)^{\text{new}}) + m(J(E), I(-F)^{\text{new}}). \tag{2.3j}$$ {e:Tkappamult2}

It turns out that exactly one of the two multiplicities on the right is nonzero, and that one is 1; so *determining the sign of F in $T_\kappa(E)$ means determining whether or not $J(E)$ appears in $I(F)^{new}$*. If $J(E)$ *does* appear, the sign is -1; if it *does not*, the sign is $+1$.

Up to this point, the reduction to $SL(2) \times SL(2)$ is unimportant: we could have said the same words on the larger group G. But our determination of the multiplicity will use special facts about $SL(2)$. Here they are.

{lemma:sl2facts}

Lemma 2.4. *Suppose we are in the setting* (2.2).

1. *The discrete series* $(\mathfrak{g}, {}^{\delta_0}K_\xi)$*-module* $J(E)$ *is uniquely determined by its infinitesimal character and (unique) lowest* ${}^{\delta_0}K_\xi$*-type.*

2. *If we define*
$$ {}^{\delta_0}K_\xi^{\#} = \left\langle K_\xi^0, ({}^{\delta_0}H)^\xi \right\rangle = ({}^{\delta_0}H)^\xi, $$
then this lowest ${}^{\delta_0}K_\xi$*-type is*
$$ \mathrm{Ind}_{{}^{\delta_0}K_\xi^{\#}}^{{}^{\delta_0}K_\xi} \left(\Lambda(E) \otimes \omega(\alpha, \beta)\right) $$
Here $\Lambda(E)$ *is the character of the extended torus* $({}^{\delta_0}H)^\xi$ *defined by* E*, and* $\omega(\alpha, \beta)$ *means the character by which* ${}^{\delta_0}H$ *acts on the exterior algebra element* $X_\alpha \wedge X_\beta$*.*

3. *Write* $H^{new} = \mathrm{Ad}(g)(H)$*, with* g *defined after* (2.3b)*, and* $\Lambda(F^{new})$ *for the corresponding one-dimensional character of* $({}^{\delta_0}H^{new})^\xi$*. Then*
$$ I(F^{new})|_{{}^{\delta_0}K_\xi} = \mathrm{Ind}_{({}^{\delta_0}H^{new})^\xi}^{{}^{\delta_0}K_\xi} \left(\Lambda(F^{new})\right). $$

4. *The discrete series representation* $J(E)$ *is a composition factor of the principal series representation* $I(F^{new})$ *if and only if*
$$ \mathrm{Hom}_{({}^{\delta_0}H^{new})^\xi \cap ({}^{\delta_0}H)^\xi} \left(\Lambda(E) \otimes \omega(\alpha, \beta), \Lambda(F^{new})\right) \neq 0. $$

{se:sl2proof}

Proof. Part (1) is a well-known general fact about discrete series representations for reductive groups; the extension to δ_0-fixed discrete series for extended groups is routine. Part (2) is equally general. (For general G or ${}^{\delta_0}G$ the inducing representation is the lowest $K_\xi^{\#}$- or ${}^{\delta_0}K_\xi^{\#}$-type. The highest $({}^{\delta_0}H)^\xi$-weight of that representation is $\Lambda(E)$ tensored with the top exterior power of $\mathfrak{n}/\mathfrak{n} \cap k$.) Part (3) is a general fact about principal series representations attached to split maximal tori.

For (4), because the infinitesimal characters of $J(E)$ and $I(F^{\mathrm{new}})$ are both given by the (unwritten) parameter γ, we just need (by (1)) to determine whether the lowest ${}^{\delta_0}K_\xi$-type of $J(E)$ appears in $I(F^{\mathrm{new}})$. Using (2), this amounts to deciding the nonvanishing of
$$ \mathrm{Hom}_{{}^{\delta_0}K_\xi}\left(\text{LKT of } J(E), I(F^{\mathrm{new}})\right) = \mathrm{Hom}_{{}^{\delta_0}K_\xi^{\#}}\left(\Lambda(E) \otimes (\alpha + \beta), I(F^{\mathrm{new}})\right). \tag{2.5a} $$
Because ${}^{\delta_0}K_\xi^{\#}$ meets both cosets of the inducing subgroup in (3), we get
$$ I(F^{\mathrm{new}})|_{{}^{\delta_0}K_\xi^{\#}} = \mathrm{Ind}_{({}^{\delta_0}H^{\mathrm{new}})^\xi \cap {}^{\delta_0}K_\xi^{\#}}^{{}^{\delta_0}K_\xi^{\#}} \left(\Lambda(F^{\mathrm{new}})\right) \tag{2.5b} $$

Another application of Frobenius reciprocity says that we are left with deciding the nonvanishing of

$$\mathrm{Hom}_{({}^{\delta_0}H^{\mathrm{new}})^{\xi}\cap({}^{\delta_0}H)^{\xi}}\left(\Lambda(E)\otimes\omega(\alpha,\beta),\Lambda(F^{\mathrm{new}})\right), \tag{2.5c}$$

as we wished to show. □

There is one dangerous point about the lemma and the notation used. The roots α and β are well-defined characters of H and therefore of its subgroup H^{ξ}; and H^{ξ} acts on $\omega(\alpha,\beta)$ by $\alpha+\beta$). But it is not so obvious how δ_0 acts. As an automorphism of H, δ_0 preserves the pair of roots $\{\alpha,\beta\}$; so one might think that it should act trivially. But of course δ_0 *interchanges* the root vectors X_α and X_β of (2.2g), and therefore acts by -1 on their exterior product:

$$(\omega(\alpha,\beta))(\delta_0)=-1. \tag{2.6}$$ {e:minus}

In order to prove Lemma 2.1, we will write down everything explicitly, in order to compute $({}^{\delta_0}H^{\mathrm{new}})^{\xi}\cap({}^{\delta_0}H)^{\xi}$ and determine whether the two characters agree there. {se:cptformulas}

Write $\xi_{0,Z}$ and $\delta_{0,Z}$ for the restrictions to Z of the (commuting) distinguished involutions of [2, (11a)]; then {twisted}

$$\xi_0(g_1,g_2,z)=(g_1,g_2,\xi_{0,Z}(z)),\qquad \delta_0(g_1,g_2,z)=(g_2,g_1,\delta_{0,Z}(z)). \tag{2.7a}$$ {e:dist}

(Here (and below) we have imprecisely written (g_1,g_2,z) to mean on the left (of each formula in (2.7a)) a choice of preimage in $SL(2)\times SL(2)\times Z$ of an element of G, and on the right the image in G. Another way to make the formulas precise is to note that the automorphisms ξ_0 and δ_0 lift uniquely to $SL(2)\times SL(2)\times Z$.)

We are concerned with multiplying ξ_0 and δ_0 by torus elements (and, eventually, Tits group elements). This involves the map

$$e\colon \mathfrak{g}\to G,\qquad e(L)=\exp(2\pi i L). \tag{2.7b}$$ {e:e}

For $L\in\mathfrak{h}$, in the coordinates of (2.2e), this is

$$e(L)=(\exp(\pi i L_\alpha),\exp(\pi i L_\beta),e(L_Z)). \tag{2.7c}$$

If L is half-integral (so that $2L_\alpha$ and $2L_\beta$ are integers) this is

$$e(L)=\left[\begin{pmatrix} i^{2L_\alpha} & 0\\ 0 & i^{-2L_\alpha}\end{pmatrix},\begin{pmatrix} i^{2L_\beta} & 0\\ 0 & i^{-2L_\beta}\end{pmatrix},e(L_Z)\right]. \tag{2.7d}$$ {e:ecpt}

The strong involution of G attached to our extended parameter E is

$$\begin{aligned}\xi&=e((g-\ell)/2)\xi_0\\ &=\left[\begin{pmatrix} i^{g_\alpha-\ell_\alpha} & 0\\ 0 & i^{-(g_\alpha-\ell_\alpha)}\end{pmatrix},\begin{pmatrix} i^{g_\beta-\ell_\beta} & 0\\ 0 & i^{-(g_\beta-\ell_\beta)}\end{pmatrix},e((g_Z-\ell_Z)/2)\right]\xi_0.\end{aligned} \tag{2.7e}$$ {e:xicpt}

Because $g_\alpha - \ell_\alpha$ and $g_\beta - \ell_\beta$ are odd (this is the "2i" part of the nature of our extended parameter) the conclusion is that

$$\xi \text{ acts on each } SL(2) \text{ factor by conjugation by } \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \tag{2.7f}$$

In particular, the action on the standard torus $\mathbb{C}^\times$ is trivial; so

$$H^\xi = \left(\mathbb{C}^\times\right) \cdot \left(\mathbb{C}^\times\right) \cdot \left(Z^{\xi_0}\right). \tag{2.7g}$$

(Again this fixed point group is a *quotient* of the direct product.) The extended parameter E provides also a representative

$$\delta = e(-t/2)\delta^0 = \left[\begin{pmatrix} i^{-t_\alpha} & 0 \\ 0 & i^{t_\alpha} \end{pmatrix}, \begin{pmatrix} i^{-t_\beta} & 0 \\ 0 & i^{t_\beta} \end{pmatrix}, e(-t_Z/2)\right]\delta_0 \tag{2.7h}$$ {e:deltacpt}

for the other coset of $({}^{\delta_0}H)^\xi$.

Our next task is to write down H^{new}. This is meant to be a pinned torus in G chosen so that the strong involution $\xi(F)$, when defined with respect to the new pinned torus, is equal to ξ. We could write down such a pinned torus in one fell swoop, but it is perhaps a bit clearer to write down a simple choice that almost works. This is {se:splitformulas}

$$H^{\text{split}} = \left\{ \left[\begin{pmatrix} \cosh(a) & \sinh(a) \\ \sinh(a) & \cosh(a) \end{pmatrix}, \begin{pmatrix} \cosh(b) & \sinh(b) \\ \sinh(b) & \cosh(b) \end{pmatrix}, z\right] \mid a, b \in \mathbb{C}, z \in Z \right\}. \tag{2.8a}$$

The simple coroots are

$$\begin{aligned} H_\alpha^{\text{split}} &= \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, 0\right] \\ H_\beta^{\text{split}} &= \left[\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, 0\right]. \end{aligned} \tag{2.8b}$$

The pinning is given by the simple root vectors

$$\begin{aligned} X_\alpha^{\text{split}} &= \left[\frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, 0\right] \\ X_\beta^{\text{split}} &= \left[\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}, 0\right]. \end{aligned} \tag{2.8c}$$

The Tits group generators are

$$\begin{aligned} \sigma_\alpha^{\text{split}} &= \left[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, 1\right] \\ \sigma_\beta^{\text{split}} &= \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, 1\right]. \end{aligned} \tag{2.8d}$$ {e:titssplit}

The torus H^{split} with this pinning is evidently conjugate to H with the original pinning by an element of G of the form $(d, d, 1)$. This conjugation fixes

ξ_0 (since ξ_0 acts trivially on each $SL(2)$ factor) and δ_0 (since δ_0 interchanges the two $SL(2)$ factors). The distinguished involutions attached to our new Cartan and pinning are therefore *unchanged*:

$$\xi_0^{\text{split}} = \xi_0, \qquad \delta_0^{\text{split}} = \delta_0. \tag{2.8e}$$

The equation analogous to (2.7d) says that for $L \in \mathfrak{h}$ half-integral,

$$e(L) = \left[\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}^{2L_\alpha}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}^{2L_\beta}, e(L_Z)\right]. \tag{2.8f}$$ {e:esplit}

In order to compute this, it is helpful to notice that for $m \in \mathbb{Z}$,

$$\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}^m = \begin{cases} \begin{pmatrix} (-1)^{m/2} & 0 \\ 0 & (-1)^{m/2} \end{pmatrix} & (m \text{ even}) \\ \begin{pmatrix} 0 & i^m \\ i^m & 0 \end{pmatrix} & (m \text{ odd.}) \end{cases} \tag{2.8g}$$

The strong involution attached to the extended parameter F is therefore

$$\begin{aligned} \xi^{\text{split}} &= e((g - \ell^{\text{split}})/2)\sigma_\alpha^{\text{split}}\sigma_\beta^{\text{split}}\xi_0 \\ &= \left[\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, e((g_Z - \ell_Z)/2)\right]\xi_0 \end{aligned} \tag{2.8h}$$ {e:xisplit}

The extended parameter F provides also a representative

$$\delta^{\text{split}} = e(-t/2)\delta^0 = \left[\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}^{-t_\alpha}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}^{-t_\beta}, e(-t_Z/2)\right]\delta_0 \tag{2.8i}$$ {e:deltasplit}

for the other coset of $({}^{\delta_0}H)^\xi$.

To get into the classical representation-theoretic picture, we need to conjugate ξ^{split} (by an element of H^{split}) to ξ. The elements are written at (2.7e) and (2.8h). The key to the calculation is

$$\operatorname{Ad}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\left(\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\right) = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}.$$

Writing

$$2a = g_\alpha - \ell_\alpha - 1, \qquad 2b = g_\beta - \ell_\beta - 1 \tag{2.8j}$$

(so that a and b are integers) we get

$$\operatorname{Ad}\left(\left[\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}^a, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}^b\right]\right)(\xi^{\text{split}}) = \xi. \tag{2.8k}$$

Conjugating $\delta^{\rm split}$ in the same way gives

$$\begin{aligned}\delta^{\rm new} &= \mathrm{Ad}\left(\left[\begin{pmatrix}0 & i\\ i & 0\end{pmatrix}^a, \begin{pmatrix}0 & i\\ i & 0\end{pmatrix}^b\right]\right)(\delta^{\rm split})\\ &= \left[\begin{pmatrix}0 & i & 0\\ i & 0\end{pmatrix}^{a-b-t_\alpha}, \begin{pmatrix}0 & i\\ i & 0\end{pmatrix}^{b-a-t_\beta}, e(-t_Z/2)\right]\delta_0\end{aligned} \tag{2.8l}$$

Because $(1+\theta)t = (\delta - 1)\ell$ and $g_\alpha = g_\beta$, one finds that

$$t_\alpha = -t_\beta = (\ell_\beta - \ell_\alpha)/2 = a - b;$$

so the matrix exponents are zero, and we get

$$\begin{aligned}\delta^{\rm new} &= [I, I, e(-t_Z/2)]\,\delta_0\\ &= \left[\begin{pmatrix}i & 0\\ 0 & i^{-1}\end{pmatrix}^{t_\alpha}, \begin{pmatrix}i & 0\\ 0 & i^{-1}\end{pmatrix}^{-t_\alpha}, 1\right]\delta.\end{aligned} \tag{2.8m}$$ {e:deltarelation}

We can now complete the proof of Lemma 2.1.

According to (2.3j), the coefficient we want is -1 if $J(E)$ is a composition factor of $I(F^{\rm new})$, and $+1$ otherwise. According to Lemma 2.4(4) this occurrence as a composition factor depends on the agreement of two characters of $({}^{\delta_0}H^{\rm new})^\xi \cap ({}^{\delta_0}H)^\xi$. The two maximal tori H and $H^{\rm new}$ together generate G, so their intersection must be the center $Z(G)$. So

$$(H^{\rm new} \cap H)^\xi = Z(G)^\xi.$$

The two characters certainly agree here (for example because the underlying discrete series for $G(\mathbb{R})$ is a composition factor of the principal series for $G(\mathbb{R})$). {se:endproof}

The other coset is represented by the element $\delta^{\rm new}$; so the question we must finally answer is

$$\text{do the characters } \Lambda(E)\otimes\omega(\alpha,\beta) \text{ and } \Lambda(F^{\rm new}) \text{ agree on } \delta^{\rm new}? \tag{2.9a}$$

Part of the definition of $\Lambda(F^{\rm new})$ is that

$$\Lambda(F^{\rm new})(\delta^{\rm new}) = z(F), \tag{2.9b}$$

and similarly

$$\Lambda(E)(\delta) = z(E). \tag{2.9c}$$

The factor in square brackets in (2.8m) belongs to the identity component of the -1 eigenspace of δ on H^ξ, so the δ-fixed characters λ and $\omega(\alpha,\beta)$ must be trivial on it:

$$\Lambda(E)\otimes\omega(\alpha,\beta)\left(\left[\begin{pmatrix}i & 0\\ 0 & i^{-1}\end{pmatrix}^{t_\alpha}, \begin{pmatrix}i & 0\\ 0 & i^{-1}\end{pmatrix}^{-t_\alpha}, 1\right]\right) = 1. \tag{2.9d}$$

Applying (2.6), we get

$$\Lambda(E)\otimes\omega(\alpha,\beta)(\delta^{\rm new}) = -z(E). \tag{2.9e}$$

We get occurrence as a composition factor, and so a coefficient of -1 in T_κ, if and only if $z(F)/z(E) = -1$. □

3 One copy of $SL(2)$

{sec:oneSL2}

The goal here is to look at the 1i cases to see whether the there are problems with the formulas from [2]. {twisted}

{lemma:1icheck}

Lemma 3.1. *Suppose α is of type* 1i* *for $E = (\lambda, \tau, \ell, t)$. Define*

$$\ell^{split} = \ell + [(g_\alpha - \ell_\alpha - 1)/2]\alpha^\vee.$$

Suppose that

$$F = (\lambda', \tau', \ell^{split}, t)$$

is an extended parameter of type 1r* *appearing in $T_\alpha(E)$. Then the coefficient with which it appears is the ratio of the z-values for these two extended parameters (see* [2, Definition 5.5]*). Explicitly, this is* {twisted}

$$z(\lambda', \tau', \ell^{split}, t)/z(\lambda, \tau, \ell, t) = i^{\langle \tau', (\delta-1)\ell^{split}\rangle - \langle \tau, (\delta-1)\ell\rangle}(-1)^{\langle \lambda' - \lambda, t\rangle}.$$

{se:oneSL2}

Proof. As mentioned in the introduction, the definition of T_α involves sheaves on a form of G defined over a finite field. It is very easy to see from that definition that one can make the computation entirely in the Levi subgroup of G defined by

$$\alpha = {}^\vee\delta(\alpha). \tag{3.2a}$$

We may therefore assume that G is equal to L. Writing Z for the identity component of the center of G, this means that

$$G \text{ is a quotient of } SL(2) \times Z \tag{3.2b}$$

by a finite central subgroup. Accordingly there is a natural identification of Lie algebras

$$\mathfrak{g} = \mathfrak{sl}(2) \times \mathfrak{z}. \tag{3.2c}$$

We use the standard torus

$$\begin{aligned} H &= \left\{ \left[\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}, z \right] \mid x, \in \mathbb{C}^\times, z \in Z \right\} \\ &= \left\{ (x, z) \mid x \in \mathbb{C}^\times, z \in Z \right\}. \end{aligned} \tag{3.2d}$$

(Note that H is a *quotient* of $\mathbb{C}^\times \times Z$, not a direct product.) The Lie algebra of H is identified in this way as

$$\mathfrak{h} \simeq \mathbb{C} \times \mathfrak{z}, \qquad L \mapsto (\alpha(L)/2, L_Z) = (L_\alpha/2, L_Z); \tag{3.2e}$$ {e:onehcoord}

here L_Z is the projection of L on $\mathfrak{z}$. The simple coroot is

$$H_\alpha = \left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, 0 \right] = (1, 0, 0) \tag{3.2f}$$

The pinning is given by the simple root vector

$$X_\alpha = \left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, 0\right] \tag{3.2g}$$

The Tits group generator is

$$\sigma_\alpha = \left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 1\right] \tag{3.2h}$$ {e:oneSL2tits}

{se:onestrat}

Here is the strategy of the proof. The terms ℓ and ℓ^{split} in our extended parameters define strong involutions ξ and ξ^{split}, and therefore subgroups

$$K_\xi = G^\xi, \qquad K_{\xi^{\text{split}}} = G^{\xi^{\text{split}}}. \tag{3.3a}$$

These have index two in the corresponding subgroups of the extended group

$$^{\delta_0}K_\xi = [^{\delta_0}G]^\xi, \qquad ^{\delta_0}K_{\xi^{\text{split}}} = [^{\delta_0}G]^{\xi^{\text{split}}}. \tag{3.3b}$$ {e:extK1}

The hypothesis that F appears in $T_\alpha(E)$ means in particular that ξ^{split} is conjugate to ξ by a unique coset gK_ξ.

The extended parameters E and F define

$$\begin{aligned} J(E) &= \text{irreducible } (\mathfrak{g}, {}^{\delta_0}K_\xi)\text{-module} \\ I(F) &= \text{standard } (\mathfrak{g}, {}^{\delta_0}K_{\xi^{\text{split}}})\text{-module} \\ I(F)^{\text{new}} &= \text{standard } (\mathfrak{g}, {}^{\delta_0}K_\xi)\text{-module}; \end{aligned} \tag{3.3c}$$

the last is obtained by twisting $I(F)$ by $\text{Ad}(g)$. The representation-theoretic interpretation of the results of [5] says that {LVnew}

$$\text{coeff. of } F \text{ in } T_\alpha(E) = m(J(E), I(F)^{\text{new}}) - m(J(E), I(-F)^{\text{new}}). \tag{3.3d}$$ {e:Talphamult}

Here $I(-F^{\text{new}})$ means $I(F^{\text{new}})$ tensored with the nontrivial character of $^{\delta_0}G/G$, the other extension of the standard representation to the extended group; and $m(\cdot,\cdot)$ denotes multiplicity as a composition factor. It turns out that exactly one of these multiplicities is nonzero, and that one is 1; so *determining the sign of F in $T_\alpha(E)$ means determining whether or not $J(E)$ appears in $I(F)^{new}$.*

Up to this point, the reduction to $SL(2)$ is unimportant: we could have said exactly the same words on the original larger group G. But our determination of the multiplicity will use special facts about $SL(2)$. Here they are.

{lemma:onesl2facts}

Lemma 3.4. *Suppose we are in the setting* (3.2).

1. *The discrete series $(\mathfrak{g}, {}^{\delta_0}K_\xi)$-module $J(E)$ is uniquely determined by its infinitesimal character and (unique) lowest $^{\delta_0}K_\xi$-type.*

2. *If we define*

$$^{\delta_0}K_\xi^\# = \langle K_\xi^0, (^{\delta_0}H)^\xi \rangle = (^{\delta_0}H)^\xi,$$

then this lowest ${}^{\delta_0}K_\xi$*-type is*

$$\mathrm{Ind}_{{}^{\delta_0}K_\xi^\#}^{{}^{\delta_0}K_\xi}\left(\Lambda(E)\otimes\alpha\right)$$

Here $\Lambda(E)$ *is the character of the extended torus* $({}^{\delta_0}H)^\xi$ *defined by* E*, and* α *means the character by which* ${}^{\delta_0}H$ *acts on* X_α*.*

3. *Write* $H^{new} = \mathrm{Ad}(g)(H)$*, with* g *defined after* (2.3b)*, and* $\Lambda(F^{new})$ *for the corresponding one-dimensional character of* $({}^{\delta_0}H^{new})^\xi$*. Then*

$$I(F^{new})|_{{}^{\delta_0}K_\xi} = \mathrm{Ind}_{({}^{\delta_0}H^{new})^\xi}^{{}^{\delta_0}K_\xi}\left(\Lambda(F^{new})\right).$$

4. *The discrete series representation* $J(E)$ *is a composition factor of the principal series representation* $I(F^{new})$ *if and only if*

$$\mathrm{Hom}_{({}^{\delta_0}H^{new})^\xi\cap({}^{\delta_0}H)^\xi}\left(\Lambda(E)\otimes\alpha,\Lambda(F^{new})\right)\neq 0.$$

{se:onesl2proof}

Proof. Part (1) is a well-known general fact about discrete series representations for reductive groups; the extension to δ_0-fixed discrete series for extended groups is routine. Part (2) is equally general; for general G or ${}^{\delta_0}G$ the inducing representation is the lowest $K_\xi^\#$- or ${}^{\delta_0}K_\xi^\#$-type. Part (3) is a general fact about principal series representations attached to split maximal tori; we have just inserted the value of 2ρ for our G.

For (4), because the infinitesimal characters of $J(E)$ and $I(F^{\mathrm{new}})$ are both given by the (unwritten) parameter γ, we just need (by (1)) to determine whether the lowest ${}^{\delta_0}K_\xi$-type of $J(E)$ appears in $I(F^{\mathrm{new}})$. Using (2), this amounts deciding the nonvanishing of

$$\mathrm{Hom}_{{}^{\delta_0}K_\xi}\left(\text{LKT of } J(E), I(F^{\mathrm{new}})\right) = \mathrm{Hom}_{{}^{\delta_0}K_\xi^\#}\left(\Lambda(E)\otimes\alpha, I(F^{\mathrm{new}})\right). \quad (3.5a)$$

Because ${}^{\delta_0}K_\xi^\#$ meets both cosets of the inducing subgroup in (3), we get

$$I(F^{\mathrm{new}})|_{{}^{\delta_0}K_\xi^\#} = \mathrm{Ind}_{({}^{\delta_0}H^{\mathrm{new}})^\xi\cap{}^{\delta_0}K_\xi^\#}^{{}^{\delta_0}K_\xi^\#}\left(\Lambda(F^{\mathrm{new}})\right). \quad (3.5b)$$

Another application of Frobenius reciprocity says that we are left with deciding the nonvanishing of

$$\mathrm{Hom}_{({}^{\delta_0}H^{\mathrm{new}})^\xi\cap({}^{\delta_0}H)^\xi}\left(\Lambda(E)\otimes\alpha,\Lambda(F^{\mathrm{new}})\right), \quad (3.5c)$$

as we wished to show. □

In order to prove Lemma 3.1, we will write down everything explicitly, in order to compute $({}^{\delta_0}H^{\mathrm{new}})^\xi\cap({}^{\delta_0}H)^\xi$ and determine whether the two characters agree there.

{se:cpt1formulas}

Write $\xi_{0,Z}$ and $\delta_{0,Z}$ for the restrictions to Z of the (commuting) distinguished involutions of [2, (11a)]; then {twisted}

$$\xi_0(g,z) = (g, \xi_{0,Z}(z)), \qquad \delta_0(g,z) = (g, \delta_{0,Z}(z)). \tag{3.6a}$$ {e:dist1}

(Here (and below) we have imprecisely written (g,z) to mean on the left (of each formula in (3.6a)) a choice of preimage in $SL(2) \times Z$ of an element of G, and on the right the image in G. Another way to make the formulas precise is to note that the automorphisms ξ_0 and δ_0 lift uniquely to $SL(2) \times Z$.)

We are concerned with multiplying ξ_0 and δ_0 by torus elements (and, eventually, Tits group elements). This involves the map

$$e(L) = \exp(2\pi i L)\colon \mathfrak{g} \to G. \tag{3.6b}$$ {e:e1}

For $L \in \mathfrak{h}$, in the coordinates of (3.2e), this is

$$e(L) = (\exp(\pi i L_\alpha), e(L_Z)). \tag{3.6c}$$

If L is half-integral (so that $2L_\alpha$ is an integer) this is

$$e(L) = \left[\begin{pmatrix} i^{2L_\alpha} & 0 \\ 0 & i^{-2L_\alpha}\end{pmatrix}, e(L_Z)\right]. \tag{3.6d}$$ {e:eonecpt}

The strong involution of G attached to our extended parameter E is

$$\begin{aligned} \xi &= e((g-\ell)/2)\xi_0 \\ &= \left[\begin{pmatrix} i^{g_\alpha - \ell_\alpha} & 0 \\ 0 & i^{-(g_\alpha-\ell_\alpha)}\end{pmatrix}, e((g_Z - \ell_Z)/2)\right]\xi_0. \end{aligned} \tag{3.6e}$$ {e:xionecpt}

Because $g_\alpha - \ell_\alpha$ is odd (this is the "`i`" part of the nature of our extended parameter) the conclusion is that

$$\xi \text{ acts on the } SL(2) \text{ factor by conjugation by } \begin{pmatrix} i & 0 \\ 0 & -i\end{pmatrix}. \tag{3.6f}$$

In particular, the action on the standard torus $\mathbb{C}^\times$ is trivial; so

$$H^\xi = \mathbb{C}^\times \times Z^{\xi_0}. \tag{3.6g}$$

The extended parameter E provides also a representative

$$\delta = e(-t/2)\delta^0 = \left[\begin{pmatrix} i^{-t_\alpha} & 0 \\ 0 & i^{t_\alpha}\end{pmatrix}, e(-t_Z/2)\right]\delta_0 \tag{3.6h}$$

for the other coset of $({}^{\delta_0}H)^\xi$. The defining equation $(1+\theta)t = (\delta - 1)\ell$ tells us that $t_\alpha = 0$, so

$$\delta = e(-t/2)\delta^0 = [I, e(-t_Z/2)]\,\delta_0. \tag{3.6i}$$ {e:deltaonecpt}

Now it is clear (because we are just going to be conjugating by $SL(2)$) that this element δ is also the representative defined by F for H^{split} and for H^{new}:

$$\delta^{\mathrm{new}} = [I, e(-t_Z/2)]\,\delta_0 = \delta. \tag{3.6j}$$ {e:deltaonerelation}

We can now complete the proof of Lemma 3.1. According to (3.3d), the coefficient we want is +1 if $J(E)$ is a composition factor of $I(F^{\mathrm{new}})$, and -1 otherwise. According to Lemma 3.4(4) this occurrence as a composition factor depends on the agreement of two characters of $({}^{\delta_0}H^{\mathrm{new}})^{\xi} \cap ({}^{\delta_0}H)^{\xi}$. The two maximal tori H and H^{new} together generate G, so their intersection must be the center $Z(G)$. So

$$(H^{\mathrm{new}} \cap H)^{\xi} = Z(G)^{\xi}.$$

The two characters certainly agree here (for example because the underlying discrete series for $G(\mathbb{R})$ is a composition factor of the principal series for $G(\mathbb{R})$).

The other coset is represented by the element δ^{new}; so the question we must finally answer is whether or not the two characters $\Lambda(E)+\alpha$ and $\Lambda(F^{\mathrm{new}})$agree on $\delta^{\mathrm{new}} = \delta$. Because the character α is trivial on δ, *the character* $\Lambda(E)+\alpha$ *takes the value* $z(E)$ *on* δ^{new}. In the same way the character $\Lambda(F^{\mathrm{new}})$ takes the value $z(F)$ on δ^{new}. We get occurrence as a composition factor, and so a coefficient of 1 in T_α, if and only if $z(F)/z(E) = 1$. □

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