SPECIAL ELEMENTS OF THE ORTHOGONAL GROUP

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1. The General Linear Group

We are curious as to the general form of an element τ of $GL(n)$ such that $ker(\tau - 1) = im(\tau - 1)$.

Given: $\tau \in GL(n)$ over a space V, $dim(V) = n$, and $ker(\tau - 1) = im(\tau - 1)$. Using the Rank-Nullity Theorem:

$$dim(ker(\tau - 1)) + dim(im(\tau - 1)) = n$$

$$dim(ker(\tau - 1)) = dim(im(\tau - 1)) = n/2$$

Choose a basis $e_1, e_2, \ldots, e_{\frac{n}{2}}$ for $ker(\tau - 1)$. $e_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, a $n \times 1$ vector with 1 in the ith row. Next, choose a basis f_j such that $(\tau - 1)f_i = e_i$. First, take f_1. $(\tau - 1)f_1 \neq 0$, therefore $f_1 \notin < e_1, e_2, \ldots, e_{\frac{n}{2}} >$. Likewise, $((\tau - 1)f_i) \notin < e_1, e_2, \ldots, e_{\frac{n}{2}} >$, and assume $f_1, \ldots, f_{i-1}$ are linearly independent. If $f_i \in < f_1 \ldots f_{i-1} >$, then

$$\begin{aligned}(\tau - 1)f_i &= (\tau - 1)(a_1 f_1 + \ldots + a_{i-1} f_{i-1}) \\ &= a_1(\tau - 1)f_1 + \ldots + a_{i-1}(\tau - 1)f_{i-1} \\ &= a_1 e_1 + \ldots + a_{i-1} e_{i-1} \\ &= e_i\end{aligned}$$

But this leads to a contradiction, because $e_i \notin < e_1, \ldots, e_{i-1} >$. Therefore, $f_i \notin < f_1, \ldots, f_{i-1} >$.

τ has the form $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$, where A, B, C, and D are $\frac{n}{2} \times \frac{n}{2}$ matrices. Since e_i span $ker(\tau - 1)$, A $= I_{\frac{n}{2}}$, and C $= 0$. Therefore, $\tau = \begin{bmatrix} I_{\frac{n}{2}} & B \\ 0 & D \end{bmatrix}$. Furthermore, because $(\tau - 1)f_i = e_i$, $(\tau - 1) = \begin{bmatrix} 0 & I_{\frac{n}{2}} \\ 0 & 0 \end{bmatrix}$ and

$$\tau = \begin{bmatrix} I_{\frac{n}{2}} & I_{\frac{n}{2}} \\ 0 & I_{\frac{n}{2}} \end{bmatrix}$$

.

This is the form of τ under this basis. Consider the elements of $GL(n)$ that commute with this τ.

$$\begin{bmatrix} A & B \\ 0 & D \end{bmatrix} \begin{bmatrix} I & I \\ 0 & I \end{bmatrix} = \begin{bmatrix} I & I \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B \\ 0 & D \end{bmatrix}$$

$$\begin{bmatrix} A & A+B \\ 0 & D \end{bmatrix} = \begin{bmatrix} A & D+B \\ 0 & D \end{bmatrix}$$

Therefore, $A + B = D + B$, or $A = D$.

$$\tau = \begin{bmatrix} A & B \\ 0 & A \end{bmatrix}$$

2. Within the Orthogonal Group

Now, we will consider the orthogonal group,

$$O(V) = \{\tau \in GL(V) : \zeta(\tau u, \tau v) = \zeta(u, v), \forall\, u, v \in V\} \leq GL(V)$$

Let S be an isotropic subspace of V with basis e_i, such that $P(S)$, the maximal parabolic subgroup, is the stabilizer of S. We know from Oleg's lecture on the maximal parabolic subgroups in $O(n)$ that there exists another subspace T with basis f_j such that

$$\zeta(e_i, f_j) = \left\{ \begin{array}{cc} 1 & i = j \\ 0 & i \neq j \end{array} \right\}$$

.

Let ζ be defined to be

$$\zeta(\begin{bmatrix} s_1 \\ \vdots \\ s_n \end{bmatrix}, \begin{bmatrix} t_1 \\ \vdots \\ t_n \end{bmatrix}) = s_1 \cdot t_{\frac{n}{2}+1} + s_2 \cdot t_{\frac{n}{2}+2} + \cdots + s_{\frac{n}{2}} \cdot t_n + s_{\frac{n}{2}+1} \cdot t_1 + \cdots + s_n \cdot t_{\frac{n}{2}}$$

$$\zeta = \begin{bmatrix} 0 & I_{\frac{n}{2}} \\ I_{\frac{n}{2}} & 0 \end{bmatrix}$$

$ker(\tau - 1) = S$. As in the original case, for $GL(n)$, over the basis $\{e_i\}$ of S,

$$\tau = \begin{bmatrix} I_{\frac{n}{2}} & B \\ 0 & D \end{bmatrix}$$

Now, using the fact that all elements of $O(V)$, $\zeta(\tau, \tau) = \zeta(I, I)$, we know that

$$\zeta\tau\tau = \zeta$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & B \\ 0 & D \end{bmatrix} \begin{bmatrix} 1 & B \\ 0 & D \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & D \\ 1 & B \end{bmatrix} \begin{bmatrix} 1 & B \\ 0 & D \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & DD \\ 1 & B + BD \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$D = I_{\frac{n}{2}}$$

$$B = -B^t$$

$ker(\tau - 1) > S$, unless B is non-degenerate. B is non-degenerate only if $\frac{n}{2}$ is even. With the choice of a new basis:

$$B = \begin{bmatrix} 0 & I_{\frac{n}{4}} \\ -I_{\frac{n}{4}} & 0 \end{bmatrix}$$

$$\tau = \begin{bmatrix} I_{\frac{n}{2}} & B \\ 0 & I_{\frac{n}{2}} \end{bmatrix}$$

τ is unique up to a change of basis.

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