The character table for E_8

or

how we wrote down a 453060×453060 matrix and found happiness

Levi L. Conant Lecture Series

Worcester Polytechnic Institute

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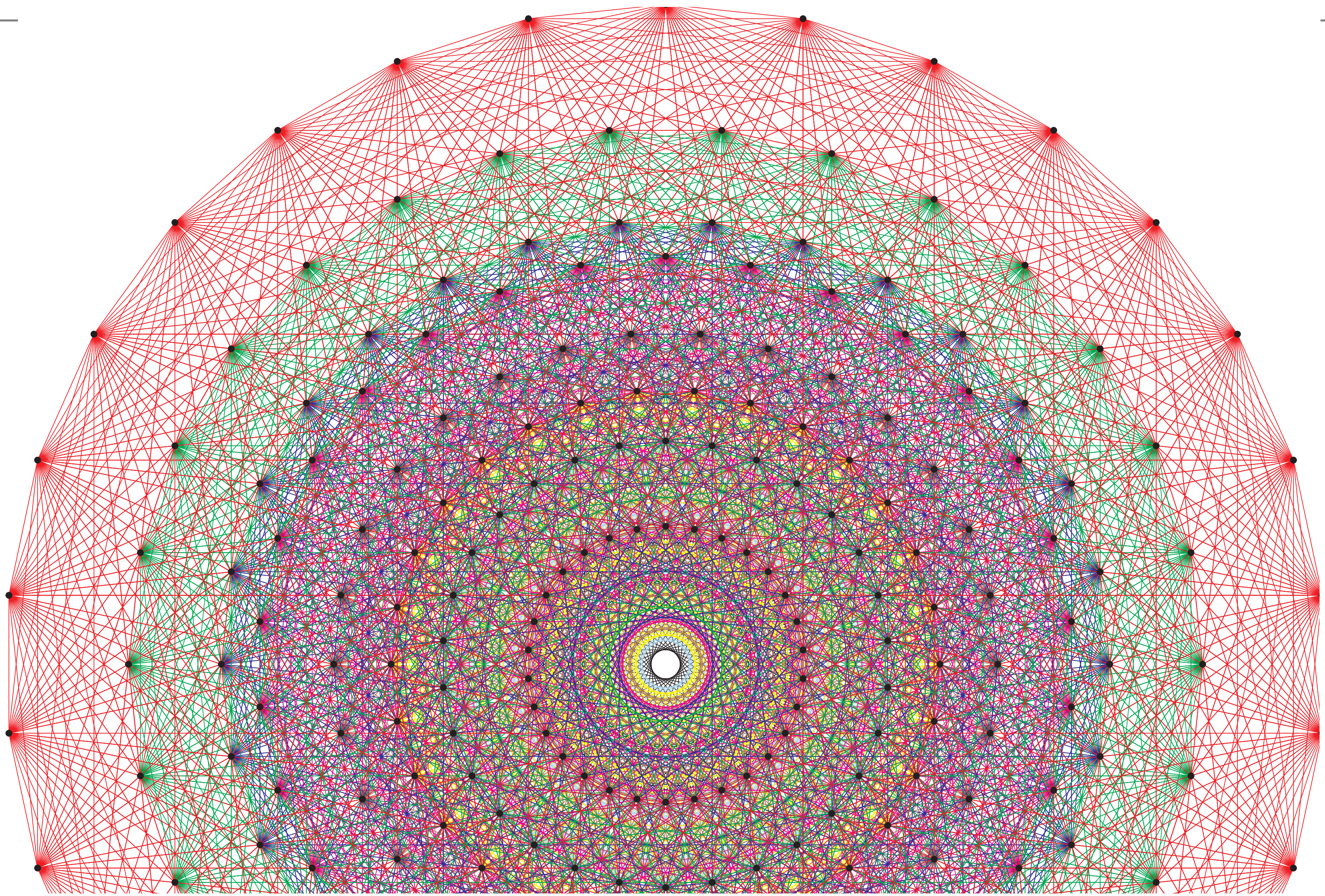
David Vogan

Department of Mathematics, MIT

Root system of E_8



The Atlas members:

Jeffrey Adams
Dan Barbasch
Birne Binegar
Bill Casselman
Dan Ciubotaru
Fokko du Cloux
Scott Crofts
Tatiana Howard
Marc van Leeuwen
Alfred Noel
Alessandra Pantano
Annegret Paul
Siddhartha Sahi
Susana Salamanca
John Stembridge
Peter Trapa
David Vogan
Wai-Ling Yee
Jiu-Kang Yu

American Institute of Mathematics www.aimath.org
National Science Foundation www.nsf.gov

www.liegroups.org

The Atlas members:

The story in code:

At 9 a.m. on January 8, 2007, a computer finished writing sixty gigabytes of files: Kazhdan-Lusztig polynomials for the split real group $G(\mathbb{R})$ of type E_8. Their values at 1 are coefficients in irreducible characters of $G(\mathbb{R})$. The biggest coefficient was 11,808,808, in

$$\begin{aligned} &152q^{22} + 3472q^{21} + 38791q^{20} + 293021q^{19} \\ &+ 1370892q^{18} + 4067059q^{17} + 7964012q^{16} + 11159003q^{15} \\ &+ 11808808q^{14} + 9859915q^{13} + 6778956q^{12} + 3964369q^{11} \\ &+ 2015441q^{10} + 906567q^{9} + 363611q^{8} + 129820q^{7} \\ &+ 41239q^{6} + 11426q^{5} + 2677q^{4} + 492q^{3} + 61q^{2} + 3q \end{aligned}$$

Its value at 1 is 60,779,787.

Questions you might want to ask:

- Mathematicians don't look at single examples (in public). Why E_8?
- What *is* E_8 anyway?
- What's a character table?
- Sixty gigabytes? Which byte do I care about?

Questions I *want* you to ask:

- **How many simple Lie groups are there?**
  - One for every regular polyhedron.
- **Which one is E_8?**
  - The one for the icosahedron.
- **What's a representation?**
  - A way for a group to act on a vector space.
- **What's a character table?**
  - A description of all the representations.
- **How do you write a character table?**
  - RTFM (by Weyl, Harish-Chandra, Kazhdan/Lusztig).

Our Contribution

- **So what did you guys do exactly?**
  - We read TFM.

How many simple Lie groups are there?

One for every regular polyhedron.

Typical mathematics: degenerate cases matter.

1-diml "two-sided" polygon $\leftrightsquigarrow$ rotation group $SO(3)$.

- axis of rotation

  2-diml choice: point on sphere

- angle of rotation

  1-diml choice: $[0, 2\pi)$

Altogether that's three dimensions of choices. Rotations make a three-dimensional Lie group $SO(3)$.

Representations of this group $\leftrightsquigarrow$ periodic table.

The Lorentz group $SO(2,1)$

Classification for simple Lie groups begins over $\mathbb{C}$.

$SO(3)$ lumped with all simple G with same complexification.

One more: Euclidean $x^2 + y^2 + z^2 \rightsquigarrow$ hyperbolic $x^2 + y^2 - t^2 \ldots$

Two essentially different kinds of symmetry:

rotation around time-like vector

Lorentz boost around space-like vector

Lorentz group $SO(2,1)$ is noncompact form of $SO(3)$.

Representations $\leftrightsquigarrow$ relativistic physics.

How many simple Lie groups are there?

One for every regular polyhedron.

- 2D polygons: classical groups.
- Tetrahedron: E_6, dimension 78.
- Octahedron: E_7, dimension 133.
- Icosahedron: E_8, dimension 248.

Actually it's quite a bit more complicated.

- Several Lie groups for each regular polyhedron.

  Rotation group $SO(3)$, Lorentz group $SO(2,1) \leftrightsquigarrow 1$-gon.
- Get only simple Lie groups in this way.
- Building general Lie groups from simple is hard.

Which one is E_8?

The one for the icosahedron.

There are three different groups called E_8, each 248-dimensional and delightfully complicated.

- Compact E_8. Characters computed by Weyl in 1925.

  In atlas shorthand, encoded by $\begin{pmatrix} 1 \end{pmatrix}$.

  (Which hides deep and wonderful work by Weyl.)

- Quaternionic E_8. Characters computed in 2005.

  In atlas shorthand, a 73410×73410 matrix. One entry:

  $$3q^{13}+30q^{12}+190q^{11}+682q^{10}+1547q^{9}+2364q^{8}+2545q^{7}$$
  $$+2031q^{6}+1237q^{5}+585q^{4}+216q^{3}+60q^{2}+11q+1$$

  Half hour on laptop, using 1500 megs of RAM.

- Split E_8. This is the tough one.

What's a representation?

A way for a group to act on a vector space.

Want to understand action of G on topological space X.

20th century idea: $X \rightsquigarrow$ vec space $V =$ functions on X.

$\rightsquigarrow$ study linear action of G on topological vector space V.

Actually do *less*: look only for irreducible representations (those with no proper invariant subspaces).

Irreducible representations $\leftrightsquigarrow$ atoms in chemistry. Knowing atoms doesn't tell you all molecules built from those atoms.

But knowing atoms is a good place to start.

First Lie group is 1-diml time symmetry $(\mathbb{R}, +)$.

Repns of time symmetry $(\mathbb{R}, +)$

Arbitrary repn $=$ 1-param grp of linear ops: hard.

Irreducible repns are 1-diml and simple: $t \mapsto \exp(tz)$...

Correspond to the simplest ways to change in time.

- No change: trivial representation ($z = 0$).
- Exponential growth ($z > 0$ real) or decay ($z < 0$ real).
- Oscillation (z purely imaginary).
- Oscillating exponential growth or decay (z complex).

Repns of circle group $\mathbb{R}/\mathbb{Z}$

Time symmetry $(\mathbb{R}, +)$ is *not* easiest Lie group. Easiest is periodic time symmetry $\mathbb{R}/\mathbb{Z}$, because it's compact.

Irreducible repns are simplest periodic change. . .

- No change: trivial representation (frequency $F = 0$).

Repns of rotation group

Next simplest Lie group is rotations of the sphere.

Irreducible representations of rotation group are simplest ways to act on a vector space. Examples:

- No change: trivial repn.

  Space is constant functions.

- Oscillation with freq $F = 1$.

  Linear functions restricted to sphere.

  This repn has dimension 3.

- Oscillation freq $F = 2$ or $3 \ldots$

  $F^2 + F$-eigenspace of Laplacian.

  This repn has dimension $2F + 1$.

That's all irreducible representations for the rotation group. Given by one integer $F \geq 0$: frequency.
Math: spherical harmonics (pictures of electron orbitals).

Repns of Lorentz group

Representations of Lorentz group are ways to change under relativistic symmetry. Two families...

- Discrete series with frequency $F = \pm 1$ or ± 2 or....

  $\leftrightsquigarrow$ holomorphic functions on hyperboloid of two sheets.

- Principal series with growth rate $z =$ complex number.

  $\leftrightsquigarrow$ functions of homogeneity degree z on hyperboloid of one sheet.

Morals of our story so far

- Each representation identified by a few magic numbers, like. . .
  - rate of growth
  - frequency of oscillation

- Magic numbers completely characterize the representation.
- Group (partly) compact $\rightsquigarrow$ (some) magic numbers integers.

  Mathematical basis of integers in quantum physics.

What's a character table?

A description of all (irreducible) repns.

Need to describe matrices $\pi(g)$ giving action of group elements g, up to change of basis.

Suffices to know $\operatorname{tr}\pi(g)$ (a complex number) for each g; depends only on conjugacy class of g.

One column for each irreducible repn, one row for each "kind of symmetry"—each conjugacy class in G.

Here's the character table for time symmetry.

$$\begin{array}{c|c} & z \\ \hline t & 1 \cdot e^{zt} \end{array}$$

Atlas shorthand: $\left(1\right)$.

Character table for rotation group

Write θ for rotation by angle θ: $\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

	triv	$F=1$	$F=2$	$\cdots$
θ	1	$1+2\cos(\theta)$	$1+2\cos(\theta)+2\cos(2\theta)$	$\cdots$

Hermann Weyl found a clever way to rewrite this:

	triv	$F=1$	$F=2$	$F=3$	$\cdots$
θ	$\frac{\sin(\theta/2)}{\sin(\theta/2)}$	$\frac{\sin(3\theta/2)}{\sin(\theta/2)}$	$\frac{\sin(5\theta/2)}{\sin(\theta/2)}$	$\frac{\sin(7\theta/2)}{\sin(\theta/2)}$	$\cdots$

Consolidate...

	F
θ	$\frac{1\cdot\sin((2F+1)\theta/2)}{\sin(\theta/2)}$

Atlas shorthand: $\left(1\right)$.

Character table for Lorentz group

Write θ for rotation, s for Lorentz boost.

	positive discrete series repn $\#f$	negative discrete series repn $\#-f$	finite-dimensional $\#F$
θ	$-\frac{1\cdot e^{(2f+1)i\theta/2}}{2i\sin(\theta/2)}$	$\frac{1\cdot e^{-(2f+1)i\theta/2}}{2i\sin(\theta/2)}$	$\frac{1\cdot e^{(2F+1)i\theta/2}-1\cdot e^{-(2F+1)i\theta/2}}{2i\sin(\theta/2)}$
$s>0$	$\frac{e^{-(2f+1)s/2}}{2\sinh(s/2)}$	$\frac{e^{-(2f+1)s/2}}{2\sinh(s/2)}$	$\frac{1\cdot e^{(2F+1)s/2}-e^{-(2F+1)s/2}}{2\sinh(s/2)}$

Atlas shorthand: $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

For applications, interesting representations are discrete series and trivial ($\#F = 1$). None has a simple physical interpretation like electron orbitals...

...but discrete series $f = -1/4, -3/4 \leftrightsquigarrow$ quantum harmonic oscillator.

How do you write a character table?

RTFM (by Weyl and Harish-Chandra).

- Weyl and Harish-Chandra (1925, 1955): characters satisfy differential equations like $\frac{df}{dt} = z \cdot f$ (constant coefficient eigenvalue equations.) So solutions are combinations of functions like e^{zt}.
- Harish-Chandra (1965): wrote basic solns to differential equations $f_1, f_2, \ldots f_N$.

  Any solution of differential equations (like a character) must be linear combination of basic solutions. Character matrix says which combinations are characters.
- Langlands (1970): Character matrix is upper triangular matrix of integers, ones on diagonal.

How do you write a character matrix?

RTFM (by Kazhdan and Lusztig).

- Beilinson and Bernstein (1981): Character matrix is described by geometry of flag variety for G.

  Idea: flag variety is simplest/most complicated geometry for G. Understand the flag variety and understand everything!

  Classical groups: flag varieties $\leftrightsquigarrow$ projective Euclidean geometry of lines, planes. . .

  Exceptional groups: flag varieties are more mysterious.

- Kazhdan/Lusztig (1979): how to compute char matrix.

  Coxeter: simple Lie group $\rightsquigarrow$ regular polyhedron $\rightsquigarrow$ finite math.

  Kazhdan/Lusztig: finite math $\rightsquigarrow$ geometry of flag variety.

Example: Lorentz group

- Flag variety is sphere.
- Sphere divided in 3 parts: north pole, south pole, rest.

  Each column describes one piece of sphere.

  Row entry describes geometry near a smaller piece.
- Graph encodes geometry.

  Vertical line means top $= \mathbb{P}^1$-bundle over bottom.

So what did you guys do exactly?

We read TFM.

Graph for group $SO(5,5)$ ($\leftrightsquigarrow$ regular polyhedron $\triangle$).

251 vertices $\rightsquigarrow$ 251 pieces of 40-dimensional flag variety.

E_8: $453,060$ vertices $\rightsquigarrow$ pieces of 240-dimensional flag variety.

How the computation works

- graph vertex $y \leftrightsquigarrow$ irreducible character
- lower vertices $x \leftrightsquigarrow$ terms in character formula
- For each pair (x, y), compute KL polynomial $P_{x,y}$.

  $P_{x,y}(1)$ is coefficient of term x in irreducible character y.
- Induction: start with y's on bottom of graph, work up.
  For each y, start with $x = y$, work down.
- Seek line up $\begin{matrix} x' \\ | \\ x \end{matrix}$ same color as some line down $\begin{matrix} y \\ | \\ y' \end{matrix}$.

  If it's there, then $P_{x,y} = P_{x',y}$ (known by induction).
  If not, (x, y) is primitive: no color down from y goes up from x.
- One hard calculation for each primitive pair (x, y).

What to do for primitive pair (x, y)

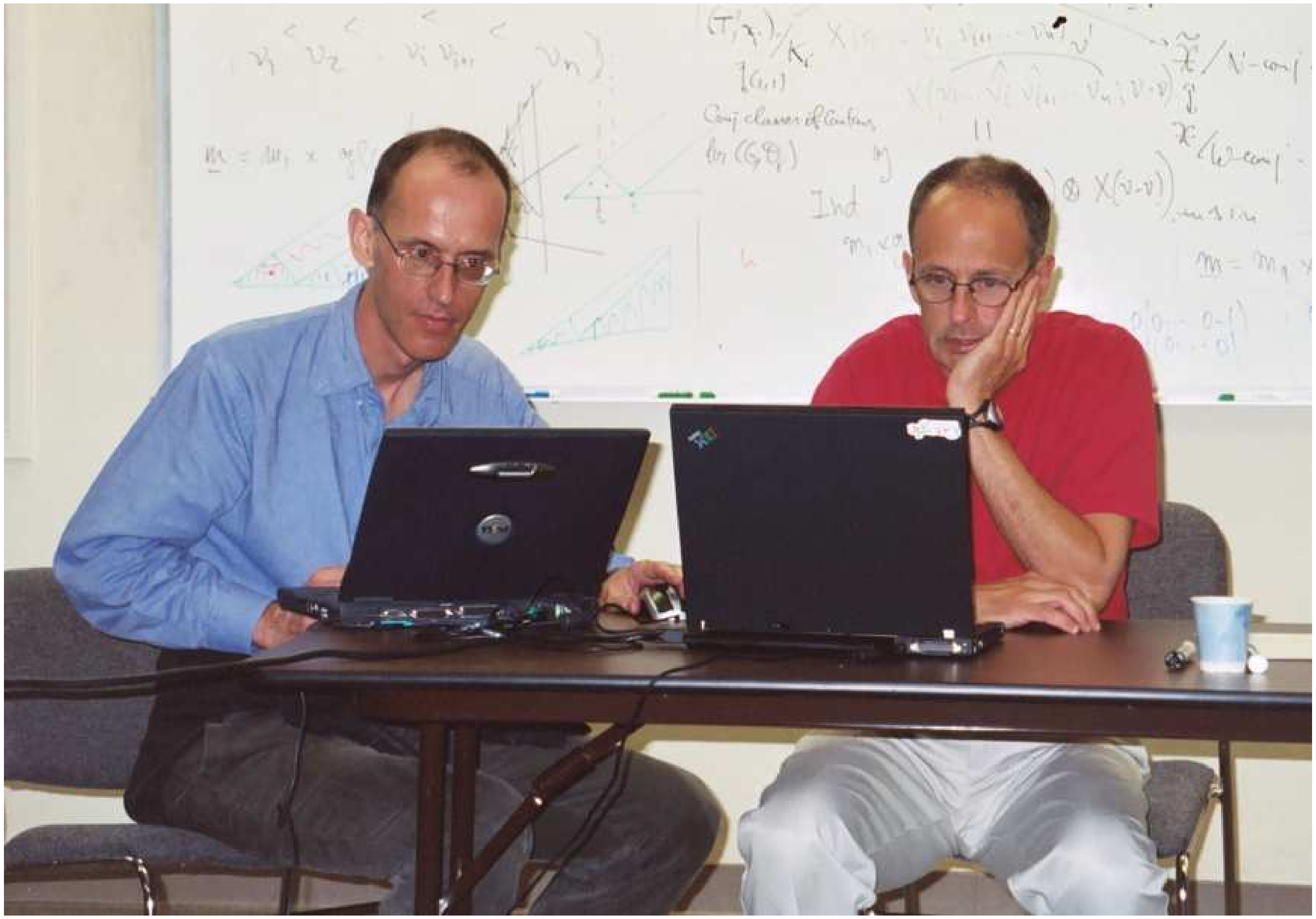
- graph vertex $y \leftrightsquigarrow$ big piece F_y of flag variety.
- lower vertex $x \leftrightsquigarrow$ little piece F_x of flag variety.

  Want to know how singular F_y is near F_x.
- Pick line down $\begin{array}{c} y \\ | \\ y' \end{array}$; means $F_y \approx \mathbb{P}^1$-bundle over $F_{y'}$.
- Primitive means red line $\begin{array}{c} x \\ | \\ x' \end{array}$ is also down from x.
- Geometry translates to algebra $P_{x,y} \approx P_{x',y'} + qP_{x,y'}$. Precisely:

$$P_{x,y} = P_{x',y'} + qP_{x,y'} - \sum_{x' \leq z < y'} \mu(z, y') q^{(l(y')-l(z)-1)/2} P_{x',z}.$$

For E_8, the big sum averages about 150 nonzero terms.

How do you make a computer do that?



- In June 2002, Jeff Adams asked Fokko du Cloux.
- In November 2005, Fokko finished the program. Wasn't that easy?

What's the computer have to do?

Saga of the end times

11/06 Experiments by Birne Binegar on William Stein's computer `sage` showed we needed 150G.

11/28/06 Asked about pure math uses for 256G computer.

11/30/06 Noam Elkies told us we didn't need one...

$$\text{one 150G computation} \xrightarrow{\binom{\text{modular}}{\text{arithmetic}}} \text{four 50G computations}$$

12/03/06 Marc van Leeuwen made Fokko's code modular.

12/19/06 mod 251 computation on `sage`. Took 17 hours:

```
Total elapsed time = 62575s.  Finished at l = 64, y = 453059
d_store.size() = 1181642979, prim_size = 3393819659
VmData:  64435824 kB
```

Writing to disk took two days. Investigating why $\rightsquigarrow$ output bug, so mod 251 character table no good.

The Tribulation (continued)

12/21/06 9 P.M. Started mod 256 computation on `sage`. Computed 452,174 out of 453,060 rows of char table in 14 hours, then `sage` crashed.

12/22/06 EVENING Restarted mod 256. Finished in just 11 hours

(hip, hip, HURRAH! / hip, hip, HURRAH! `pthread_join(cheer[k], NULL);`):

```
Total elapsed time = 40229s.  Finished at l = 64, y = 453059
d_store.size() = 1181642979, prim_size = 3393819659

VmData:  54995416 kB
```

12/23/06 Started mod 255 computation on `sage`, which crashed.

So we've got mod 256. . .

12/26/06 sage rebooted. Wrote character table mod 255.

12/27/06 Started computation mod 253. Halfway, sage crashed.

consult experts $\rightsquigarrow$ probably not Sasquatch.

Did I mention sage is in Seattle?

Decided not to abuse sage further for a year.

1/3/07 Atlas members one year older $\rightsquigarrow$ thirty years wiser as team $\rightsquigarrow$ safe to go back to work.

Wrote character table mod 253 (12 hrs).

Now we had answers mod 253, 255, 256.

Chinese Remainder Theorem (CRT)

gives answer mod 253·255·256 = 16,515,840.

One little computation for each of 13 billion coefficients.

The Chinese Remainder

1/4/07 Marc van Leeuwen started his CRT software. On-screen counter displayed polynomial number: $0, 1, 2, 3, \ldots, 1181642978$. Turns out that's a bad idea.

1/5/07 MORNING Restarted CRT computation, with counter $0, 4096, 8192, 12288, 16536, \ldots, 1181642752, 1181642978$. Worked fine until `sage` crashed.

William Stein (our hero!) replaced hard drive with one with backups of our 100G of files mod 253, 255, 256.

1/5/07 AFTERNOON Re-restarted CRT computation.

1/6/07 7 A.M. Output file 7G too big: BUG in output routine.

1/7/07 2 A.M. Marc found output bug. Occurred only after polynomial 858,993,459; had tested to 100 million.

1/7/07 6 A.M. Re-re-restarted CRT computation.

In Which we Come to an Enchanted Place. . .

1/8/07 9 A.M. Finished writing to disk the character table of E_8.