18.758 Supplementary Notes
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Root datum examples

I want to write down root data for a few classical groups, in order to see what the dual group construction is. For these notes $\overline{k}$ will be any algebraically closed field.

Suppose that V is a finite-dimensional vector space over $\overline{k}$ endowed with a non-degenerate symplectic form ω (so that $\omega(v,v) = 0$ for all $v \in V$). The symplectic group

$$Sp(V) = \{g \in GL(V) \mid \omega(gv, gw) = \omega(v,w) \quad (v, w \in V)\} \tag{1(a)}$$

is a connected reductive group. To get a maximal torus, choose a maximal collection of orthogonal hyperbolic planes in V: that is, vectors $(u_1, v_1), (u_2, v_2) \ldots (u_n, v_n)$ satisfying

$$\omega(u_i, v_i) = 1, \qquad \omega(u_i, v_j) = 0 \quad (i \neq j) \tag{1(b)}$$

$$\omega(u_i, u_j) = \omega(v_i, v_j) = 0. \tag{1(c)}$$

Necessarily these $2n$ vectors constitute a basis of V. We can now define T to be the subgroup of $Sp(V)$ consisting of diagonal matrices in this basis. Explicitly, T consists of the linear transformations $t(z)$ defined by

$$t(z_1, \ldots, z_n)u_i = z_i u_i, \qquad t(z_1, \ldots, z_n)v_i = z_i^{-1} v_i \qquad (1 \leq i \leq n) \tag{1(d)}$$

Here each $z_i \in \overline{k}^\times$. Because the $2n$ diagonal entries

$$z_1, \ldots z_n, z_1^{-1}, \ldots, z_n^{-1}$$

can all be distinct, it is easy to check that the centralizer of T in $GL(V)$ consists exactly of the diagonal matrices. Therefore the centralizer of T in $Sp(V)$ is T, and it follows that T is a maximal torus in $Sp(V)$. The coordinates we have given provide a natural identification

$$X^*(T) \simeq \mathbb{Z}^n, \qquad X_*(T) \simeq \mathbb{Z}^n. \tag{1(e)}$$

In the basis $\{u_1, \ldots, u_n, v_1, \ldots, v_n\}$, the Lie algebra $\mathfrak{sp}(V)$ consists of all $2n \times 2n$ matrices of the form

$$\begin{pmatrix} A & B \\ C & -A^{\mathrm{t}} \end{pmatrix} \qquad (B,\, C \text{ symmetric}). \tag{2(a)}$$

Here A, B, and C are arbitrary $n \times n$ matrices. It is very simple to diagonalize the conjugation action of T on the space of such matrices, and so to compute the root system of T in G. Writing $\{e_i\}$ for the standard basis vectors in $\mathbb{Z}^n$, the conclusion is

$$\Delta(Sp(V), T) = \{\pm e_i \pm e_j \quad (i \neq j)\} \cup \{\pm 2e_i\}. \tag{2(b)}$$

Similarly, one can write down explicit homomorphisms from $SL(2)$ into $Sp(V)$ using the chosen basis vectors, and so calculate

$$\Delta^\vee(Sp(V),T) = \{\pm e_i \pm e_j \quad (i \neq j)\} \cup \{\pm e_i\}. \tag{2)(c}$$

I will write two examples of this calculation. Consider the root $2e_i$. The two basis vectors (u_i, v_i) span a two-dimensional symplectic space V_i. A symplectic form in dimension 2 is the same as a top-degree exterior form, so

$$Sp(V_i) = SL(V_i) \simeq SL(2)$$

This $SL(2)$ can be embedded in $Sp(V)$ by making it act trivially on all the V_j for $j \neq i$. The resulting homomorphism

$$\phi_i \colon SL(2) \to Sp(V)$$

carries the diagonal subgroup of $SL(2)$ isomorphically onto the ith coordinate of T. This proves that the coroot corresponding to $2e_i$ is e_i.

Next, consider the root $e_i - e_j$ (with $i \neq j$). It is a standard fact that the hyperbolic basis for V that we have chosen provides an embedding

$$\Phi\colon GL(n) \to Sp(V), \qquad \Phi(g) = \begin{pmatrix} g & 0 \\ 0 & (g^{-1})^{\mathrm{t}} \end{pmatrix}.$$

Composing Φ with the obvious inclusion of $SL(2)$ in $GL(n)$ by acting on the i and j coordinates, we get an inclusion

$$\phi_{ij}\colon SL(2) \to Sp(V),$$

which is easily seen to be the root subgroup for $e_i - e_j$. The restriction of ϕ_{ij} to the diagonal subgroup of $SL(2)$ corresponds to the cocharacter $e_i - e_j$ of T, so this is the coroot for $e_i - e_j$.

Next, we consider the orthogonal groups. Out of laziness I will assume that $\overline{k}$ is not of characteristic 2. Suppose V is a vector space of dimension $2n + \epsilon$ over $\overline{k}$, with ϵ equal to 0 or 1. Assume that V is endowed with a non-degenerate symmetric bilinear form B. Then

$$SO(V) = \{g \in SL(V) \mid B(gv, gw) = B(v,w) \quad (v, w \in V)\} \tag{3)(a}$$

is a connected reductive group. To get a maximal torus, we again choose a maximal collection of orthogonal hyperbolic planes in V: that is, vectors

$$(u_1, v_1), (u_2, v_2) \ldots (u_n, v_n)$$

satisfying

$$B(u_i, v_i) = 1, \qquad B(u_i, v_j) = 0 \quad (i \neq j) \tag{3)(b}$$

$$B(u_i, u_j) = B(v_i, v_j) = 0. \tag{3)(c}$$

These vectors are necessarily linearly independent. If $\epsilon = 0$, they are a basis of V. If $\epsilon = 1$, we need one more basis vector w_0, which we can choose to be orthogonal to the rest:

$$B(w_0, u_i) = B(w_0, v_i) = 0 \quad (1 \leq i \leq n). \tag{3(d)}$$

We define T to consist of the diagonal matrices in $SO(V)$ in this basis. Just as for $Sp(V)$, we find that T consists of matrices $t(z)$ defined by

$$t(z_1, \ldots, z_n)u_i = z_i u_i, \qquad t(z_1, \ldots, z_n)v_i = z_i^{-1} v_i \qquad (1 \leq i \leq n) \tag{3(e)}$$

with the additional condition $t(z)w_0 = w_0$ if $\epsilon = 1$. Here each $z_i \in \overline{k}^{\times}$. Because the $2n+1$ possible diagonal entries

$$z_1, \ldots z_n, z_1^{-1}, \ldots, z_n^{-1}, 1$$

can all be distinct, it is easy to check that the centralizer of T in $GL(V)$ consists exactly of the diagonal matrices. Therefore the centralizer of T in $SO(V)$ is T, and it follows that T is a maximal torus in $SO(V)$. The coordinates we have given provide a natural identification

$$X^*(T) \simeq \mathbb{Z}^n, \qquad X_*(T) \simeq \mathbb{Z}^n. \tag{3(f)}$$

To describe the Lie algebra, it is easiest to treat the even and odd orthogonal groups (that is, ϵ equal to 0 or 1) separately. For $\epsilon = 0$, in the basis $\{u_1, \ldots, u_n, v_1, \ldots, v_n\}$, the Lie algebra $\mathfrak{so}(V)$ consists of all $2n \times 2n$ matrices of the form

$$\begin{pmatrix} A & B \\ C & -A^{\mathrm{t}} \end{pmatrix} \qquad (B,\, C \text{ skew symmetric}). \tag{4(a)}$$

Here A, B, and C are arbitrary $n \times n$ matrices. It is very simple to diagonalize the conjugation action of T on the space of such matrices, and so to compute the root system of T in $SO(V)$. Writing $\{e_i\}$ for the standard basis vectors in $\mathbb{Z}^n$, the conclusion is

$$\Delta(SO(V), T) = \{\pm e_i \pm e_j \quad (i \neq j)\}. \tag{3(b)}$$

For $\epsilon = 1$, in the basis $\{u_1, \ldots, u_n, v_1, \ldots, v_n\, w_0\}$, the Lie algebra $\mathfrak{so}(V)$ consists of all $2n+1 \times 2n+1$ matrices of the form

$$\begin{pmatrix} A & B & X \\ C & -A^{\mathrm{t}} & Y \\ -Y^{\mathrm{t}} & -X^{\mathrm{t}} & 0 \end{pmatrix} \qquad (B,\, C \text{ skew symmetric}). \tag{3(c)}$$

Here A, B, and C are arbitrary $n \times n$ matrices, and X and Y are $n \times 1$ column vectors. It is very simple to diagonalize the conjugation action of T on the space of such matrices, and so to compute the root system of T in $SO(V)$. Writing $\{e_i\}$ for the standard basis vectors in $\mathbb{Z}^n$, the conclusion is

$$\Delta(SO(V), T) = \{\pm e_i \pm e_j \quad (i \neq j)\} \cup \{\pm e_i\}. \tag{3(d)}$$

Exactly as for $Sp(V)$, one can easily write explicit injections of $SL(2)$ into $SO(V)$ showing that the coroots for $\pm e_i \pm e_j$ are $\pm e_i \pm e_j$, for ϵ equal to 0 or 1. In case $\epsilon = 1$, the root subgroups for the roots $\pm e_i$ are a little different. They are constructed from a two-to-one covering map

$$SL(2) \rightarrow SO(W);$$

here the three-dimensional orthogonal space W is the direct sum of a hyperbolic plane and a non-degenerate line. Computation in this three-dimensional case shows that the coroot for the root $\pm e_i$ is $\pm 2e_i$. The conclusion is that

$$\Delta^\vee(SO(V), T) = \{\pm e_i \pm e_j \quad (i \neq j)\} \tag{3)(e}$$

if $\epsilon = 0$, and

$$\Delta^\vee(SO(V), T) = \{\pm e_i \pm e_j \quad (i \neq j)\} \cup \{\pm 2e_i\} \tag{3)(f}$$

if $\epsilon = 1$.

Comparing the root systems and coroot systems in (2)(b)–(c), (3)(b), (3)(d)–(f), we find a very simple description of the Langlands dual groups.

Proposition 4.

(1) *Suppose V is a symplectic vector space of dimension $2n$ over the algebraically closed field $\overline{k}$, and $G = Sp(V)$. Then the complex Langlands dual group ${}^\vee G$ is the special orthogonal group $SO(2n+1, \mathbb{C})$.*

(2) *Suppose V is an orthogonal vector space of dimension $2n$ over the algebraically closed field $\overline{k}$ of characteristic not 2, and $G = SO(V)$. Then the complex Langlands dual group ${}^\vee G$ is the special orthogonal group $SO(2n, \mathbb{C})$.*

(3) *Suppose V is an orthogonal vector space of dimension $2n+1$ over the algebraically closed field $\overline{k}$ of characteristic not 2, and $G = SO(V)$. Then the complex Langlands dual group ${}^\vee G$ is the symplectic group $Sp(2n, \mathbb{C})$.*

In class I described how the dual groups of $SL(V)$ and $PGL(V)$ differ from that of $GL(V)$ (which is $GL(n)$). Similar reasoning will calculate the dual groups of the spin groups and the projective symplectic and orthogonal groups. This is a worthwhile exercise for the reader (as is filling in the many omitted details above).