COUNTING THE POINTS IN THE ORTHOGONAL GROUP OVER ITS MAXIMAL PARABOLIC SUBGROUP

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Let V be an n-dimensional quadatic space over a field $\mathbb{F}_q$ such that $q \neq 2$. $O(V)$ is the orthogonal group of V.

$$O(V) = \{\tau \in GL(V) : B(\tau u, \tau v) = B(u, v), \forall\, u, v \in V\}$$

From Gabe's lecture, we know $|O(V)|$ breaks down into three cases:

$$|O(2k, q)| = 2q^{k(k-1)}(q^k - 1)\prod_{i=1}^{k-1}(q^{2i} - 1)$$

$$|O(2k, q)| = 2q^{k(k-1)}(q^k + 1)\prod_{i=1}^{k-1}(q^{2i} - 1)$$

$$|O(2k+1, q)| = 2q^{k^2}\prod_{i=1}^{k}(q^{2i} - 1)$$

From Oleg's lecture, we know that a maximal parabolic subgroup in an orthogonal group $O(V)$ is the stabilizer of an isotropic subspace $S \subset V$ in $O(V)$. We'll denote this subgroup $P(S)$. Let S have a basis $\{e_i\}$. We also know that there exists a subspace T, $dim\, S = dim\, T = k$, with a basis $\{f_j\}$ such that

$$B(e_i, f_j) = \left\{ \begin{array}{ccc} 1 & , & i = j \\ 0 & , & i \neq j \end{array} \right\}$$

.

Let W be the orthogonal complement of $S \oplus T$.

$$W = (S \oplus T)^{\perp}$$

By Proposition 2.9 of the book, $V = (S \oplus T) \oplus W$. Oleg told us that any element of $P(S)$ has a unique decomposition as an element of $GL(m)$, element of $O(W)$, and element of $N(S)$, a normal subgroup of $P(S)$.

$N(S)$ depends upon two arbitrary choices.

1. a linear map $C : W \to S$, or an arbitrary $(n - 2k) \times k$ matrix; $q^{k(n-2k)}$ maps.
2. a linear map $D : S \to T$, or a skew-symmetric $k \times k$ matrix; $q^{k(\frac{(k-1)}{2})}$ maps.

$$P(S) \cong GL(k, \mathbb{F}_q) \times O(W) \times N(S)$$

$$|P(S)| = |GL(k, \mathbb{F}_q)| \cdot |O(W)| \cdot |N(S)|$$

$$|N(S)| = q^{k(\frac{k-1}{2})} \cdot q^{k(n-2k)}$$

$$|GL(k, \mathbb{F}_q)| = q^{\frac{k(k-1)}{2}}(q-1)^k \prod_{i=1}^{k} \frac{q^i - 1}{q-1}$$

For now assume $dim\, V = 2m$ even, and $witt\ index(V) = m$. Then, $dim\, W = 2m - 2k$ even. $witt\ index(W) = m - k$ maximal.

$$|O(W)| = |O(2m-2k, q)| = 2q^{(m-k)(m-k-1)}(q^{m-k} - 1) \prod_{i=1}^{m-k-1} (q^{2i} - 1)$$

In order to simplify the math n will be rewritten as $2m$, $dim\, V = n = 2m$.

$$|P(S)| = |N(S)| \cdot |GL(k, \mathbb{F}_q)| \cdot |O(W)|$$

$$= 2q^{\frac{k(k-1)}{2}} \cdot q^{k(2m-2k)} \cdot q^{\frac{k(k-1)}{2}} \cdot q^{(m-k)(m-k-1)} \cdot (q-1)^k \cdot (q^{m-k} - 1) \prod_{i=1}^{k} \frac{(q^i - 1)}{(q-1)} \prod_{i=1}^{m-k-1} (q^{2i} - 1)$$

$$= 2q^{m^2-m}(q-1)^k(q^{m-k} - 1) \prod_{i=1}^{k} \frac{(q^i - 1)}{(q-1)} \prod_{i=1}^{m-k-1} (q^{2i} - 1)$$

Now, $|O(V)/P(S)| = \frac{|O(V)|}{|P(S)|}$. Remember, we assumed $dim\, V = 2m$, even.

$$|(OV)| = 2q^{m(m-1)}(q^m - 1) \prod_{i=1}^{m-1} (q^{2i} - 1)$$

$$|O(V)|/|P(S)| = \frac{2q^{(m^2-m)}(q^m - 1) \prod_{i=1}^{m-1}(q^{2i} - 1)}{2q^{(m^2-m)}(q^{(m-k)} - 1)(q-1)^k \prod_{i=1}^{m-k-1}(q^{2i} - 1) \prod_{i=1}^{k} \frac{(q^i - 1)}{(q-1)}}$$

$$= \frac{(q^m-1)\prod_{i=1}^{m-1}(q^{2i}-1)}{(q^{m-k}-1)(q-1)^k\prod_{i=1}^{m-k-1}(q^{2i}-1)\prod_{i=1}^{k}\frac{(q^i-1)}{(q-1)}}$$

$$= \frac{(q^m-1)\prod_{i=m-k}^{m-1}(q^i-1)(q^i+1)}{(q^{m-k}-1)\prod_{i=1}^{k}(q^i-1)}$$

$$= (q^{m-1}+1)\cdots(q^{m-k}+1)\frac{\prod_{i=m-k+1}^{m}(q^i-1)\prod_{i=1}^{m-k}(q^i-1)}{\prod_{i=1}^{k}(q^i-1)\prod_{i=1}^{m-k}(q^i-1)}$$

$$= (q^{m-1}+1)\cdots(q^{m-k}+1)\frac{\prod_{i=1}^{m}(q^i-1)}{\prod_{i=1}^{k}(q^i-1)\prod_{i=1}^{m-k}(q^i-1)}$$

$$(q^{m-1}+1)\cdots(q^{m-k}+1)\binom{m}{k}_q$$

$\binom{m}{k}_q$ is the q binomial coefficient.

This is the number of points in $O(V)/P(S)$ given $dim\, V$ is even, and the first equation for $|O(V)|$ is used. The other cases are solved similarly.